First-Year Mathematics

Problem Set 9 March 4, 2005

1. Find the curls of the following vector fields:

   (a) $\boldsymbol{V} = 2xyz^3\,\boldsymbol{i} + (x^2y^3 + 2y)\,\boldsymbol{j} + 3x^2yz^2\,\boldsymbol{k}$

   (b) $\boldsymbol{V} = 2xy\,\boldsymbol{i} + (x^2 + 2yz)\,\boldsymbol{j} + (y^2 + 1)\,\boldsymbol{k}$

   (c) $\boldsymbol{V} = -xy\,\boldsymbol{i} - yz\,\boldsymbol{j} - xz\,\boldsymbol{k}$

2. Stokes' theorem states that

$$\oint_{\partial\sigma} \boldsymbol{V} \cdot d\boldsymbol{r} = \iint_{\sigma} (\boldsymbol{\nabla} \times \boldsymbol{V}) \cdot \boldsymbol{n}\, d\sigma\,,$$

   where $\mathbf{V}$ is a vector field, $\partial\sigma$ is the bounding curve for the surface σ, and the line integral is taken in the counterclockwise direction.

   (a) Consider a right square prism of height h with a bounding curve in the x-y plane that has vertices at $(0,0,0)$, $(a,0,0)$, $(a,a,0)$, and $(0,a,0)$. Evaluate the left-hand side of Stokes' theorem around this curve for the vector field

$$\boldsymbol{V} = -xy\,\boldsymbol{i} - yz\,\boldsymbol{j} - xz\,\boldsymbol{k}\,.$$

   Ans: $\frac{1}{2}a^3$.

   (b) The prism in Part (a) has height h, so the vertices of its top face are at $(0,0,h)$, $(a,0,h)$, (a,a,h), and $(0,a,h)$. Evaluate the right-hand side of Stokes' theorem. How do you interpret the fact that the integral does not depend on h?

   (c) By direct calculation, or otherwise, determine the right-hand side of Stokes' theorem for the following surfaces, each of which have the same base curve as the prism in Part (a).

      i. The interior of the square contained within the bounding curve.

      ii. The "inverted" square prism with a bottom face that has vertices at $(0,0,-h)$, $(a,0,-h)$, $(a,a,-h)$, and $(0,a,-h)$.

      iii. A square pyramid of height h.

   Ans: (a) $\frac{1}{2}a^3$, (b) $-\frac{1}{2}a^3$, (c) $\frac{1}{2}a^3$.

3. Consider the vector field

$$\boldsymbol{V}(x, y) = -y\,\boldsymbol{i} + x\,\boldsymbol{j}\,.$$

Show that the curl of this vector field is

$$\boldsymbol{\nabla} \times \boldsymbol{V} = 2\,\boldsymbol{k}\,.$$

This vector field is shown in Fig. 7.4 of the course notes. The vectors are seen to "swirl" counter-clockwise around the origin However, $\boldsymbol{\nabla} \times \boldsymbol{V} = 2\,\boldsymbol{k}$ at *all* points. The following exercises explain the meaning of the curl in a manner similar to that used for the divergence in Classwork 7.

(a) As in Classwork 7, imagine that $\boldsymbol{V}$ is the velocity field of a fluid and that you are riding on a fluid particle. The relative velocities at your point (x_0, y_0) are computed by subtracting the instantaneous velocity $\boldsymbol{V}(x_0, y_0)$ at this point from the instantaneous velocity at all other points. This gives the relative velocity field

$$\boldsymbol{V}(x, y) - \boldsymbol{V}(x_0, y_0)\,.$$

Compute the curl of this vector field. Choose a particular reference point, say $(1, 1)$, and sketch the velocity field relative to this point and compare with your sketch of $\boldsymbol{V}$. What can you conclude about the curl of $\boldsymbol{V}$ having the same value everywhere?

(b) Consider the vector field

$$\boldsymbol{V}(x, y) = (ax + by)\,\boldsymbol{i} + (cx + dy)\,\boldsymbol{j}\,,$$

where a, b, c, and d are constants. Compute the curl of $\boldsymbol{V}$. Identify the conditions that determine whether the curl is positive, negative, or zero. As an example of a vector field with zero curl, consider

$$\boldsymbol{V}(x, y) = y\,\boldsymbol{i} + x\,\boldsymbol{j}\,.$$

Provide an interpretation for the zero curl of this vector field using the analogy in Part (a).

(c) Consider the vector field

$$\boldsymbol{V}(x, y) = y^2\,\boldsymbol{i} + x^2\,\boldsymbol{j}\,.$$

Follow the steps in Part (a) for the points $(1, 0)$, $(1, 1)$, and $(0, 1)$ and provide an interpretation of your results.