First-Year Mathematics

Solutions to Problem Set 8 February 25, 2005

1. (a) The outward normal to a sphere of radius R is calculated by taking the gradient of the equation $x^2+y^2+z^2=R^2$:

$$\boldsymbol{\nabla}(x^2+y^2+z^2) = 2x\,\boldsymbol{i} + 2y\,\boldsymbol{j} + 2z\,\boldsymbol{k}\,. \tag{1}$$

The magnitude of this vector on the sphere of radius unity is obtained from

$$\left[\boldsymbol{\nabla}(x^2+y^2+z^2)\cdot\boldsymbol{\nabla}(x^2+y^2+z^2)\right]\Bigg|_{R=1} = 4x^2+4y^2+4z^2\Bigg|_{R=1} = 4\,. \tag{2}$$

Thus,

$$\boldsymbol{n} = \frac{\boldsymbol{\nabla}(x^2+y^2+z^2)}{|\boldsymbol{\nabla}(x^2+y^2+z^2)|} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}\,. \tag{3}$$

(b) The "dot" product $\boldsymbol{V}\cdot\boldsymbol{n}$ is

$$(x\,\boldsymbol{i} + y\,\boldsymbol{j} + xyz\,\boldsymbol{k})\cdot(x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\boldsymbol{k}) = x^2+y^2+xyz^2\,. \tag{4}$$

In spherical polar coordinates on the surface of the unit sphere, we have

$$x = \sin\theta\cos\phi\,, \qquad y = \sin\theta\sin\phi\,, \qquad z=\cos\theta\,, \qquad d\sigma = \sin\theta\,d\theta\,d\phi\,. \tag{5}$$

Thus,

$$\begin{aligned} x^2+y^2+xyz^2 &= (\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \sin^2\theta\cos^2\theta\cos\phi\sin\phi \\ &= \sin^2\theta + \sin^2\theta\cos^2\theta\cos\phi\sin\phi\,, \end{aligned} \tag{6}$$

so the surface integral becomes

$$\begin{aligned} \iint \boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma &= \int_0^{2\pi} d\,\phi \int_0^{\pi} \sin\theta\,d\theta(\sin^2\theta + \cos\phi\sin\phi\sin^2\theta\cos^2\theta) \\ &= \underbrace{\int_0^{2\pi} d\,\phi}_{2\pi} \int_0^{\pi} \sin^3\theta\,d\theta + \underbrace{\int_0^{2\pi} \cos\phi\sin\phi\,d\,\phi}_{0} \int_0^{\pi} \sin^2\theta\cos^2\theta\,d\theta \\ &= 2\pi\int_0^{\pi}\sin^3\theta\,d\theta\,. \end{aligned} \tag{7}$$

This integral is straightforward to carry out:

$$\begin{aligned}
\int_0^\pi \sin^3\theta\, d\theta &= \int_0^\pi \sin\theta(1-\cos^2\theta)\, d\theta \\
&= \int_0^\pi \sin\theta\, d\theta - \int_0^\pi \sin\theta\cos^2\theta\, d\theta \\
&= -\cos\theta\Big|_0^\pi + \tfrac{1}{3}\cos^3\theta\Big|_0^\pi \\
&= 2 - \frac{2}{3} \\
&= \frac{4}{3}\,. \qquad (8)
\end{aligned}$$

The surface integral therefore evaluates to

$$\iint \boldsymbol{V}\cdot\boldsymbol{n}\, d\sigma = \frac{8}{3}\pi\,. \qquad (9)$$

(c) The divergence of $\boldsymbol{V}$ is

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = 1+1+xy = 2+xy\,. \qquad (10)$$

In spherical polar coordinates, we have

$$x = r\sin\theta\cos\phi\,, \qquad y = r\sin\theta\sin\phi\,, \qquad d\tau = r^2\sin\theta\, dr\, d\theta\, d\phi\,. \qquad (11)$$

Thus,

$$\begin{aligned}
\iint \boldsymbol{V}\cdot\boldsymbol{n}\, d\sigma &= \int_0^{2\pi} d\,\phi \int_0^\pi \sin\theta\, d\theta(2+\cos\phi\sin\phi\sin^2\theta) \\
&= 2\int_0^1 r^2\, dr \int_0^{2\pi} d\phi \int_0^\pi \sin\theta\, d\theta \\
&\quad + \int_0^1 r^4\, dr \underbrace{\int_0^{2\pi} \cos\phi\sin\phi\, d\phi}_{0} \int_0^\pi \sin^2\theta\, d\theta \\
&= 2\underbrace{\int_0^1 r^2\, dr}_{\frac{1}{3}} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^\pi \sin\theta\, d\theta}_{2} \\
&= \frac{8}{3}\pi\,, \qquad (12)
\end{aligned}$$

which agrees with Eq. (9).

2. (a) The "dot" product $\boldsymbol{V} \cdot \boldsymbol{n}$ is

$$\boldsymbol{V} \cdot \boldsymbol{n} = \left(\frac{x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}}{\sqrt{x^2+y^2+z^2}} \right) \cdot (x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k})$$

$$= \frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}}$$

$$= \sqrt{x^2+y^2+z^2}\,. \tag{13}$$

Again using spherical coordinates for the surface integral over the unit sphere, we have that $x^2+y^2+z^2=1$, so that

$$\iint \boldsymbol{V} \cdot \boldsymbol{n}\, d\sigma = \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi} \sin\theta\, d\theta}_{2} = 4\pi\,, \tag{14}$$

which is the surface area of the sphere.

(b) The divergence of $\boldsymbol{V}$ is

$$\boldsymbol{\nabla} \cdot \boldsymbol{V}$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}} \right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}} \right) + \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}} \right)\,. \tag{15}$$

The partial derivatives are evaluated as:

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}} \right) = \frac{1}{x^2+y^2+z^2}\left(\sqrt{x^2+y^2+z^2} - \frac{x^2}{\sqrt{x^2+y^2+z^2}} \right), \tag{16}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}} \right) = \frac{1}{x^2+y^2+z^2}\left(\sqrt{x^2+y^2+z^2} - \frac{y^2}{\sqrt{x^2+y^2+z^2}} \right), \tag{17}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}} \right) = \frac{1}{x^2+y^2+z^2}\left(\sqrt{x^2+y^2+z^2} - \frac{z^2}{\sqrt{x^2+y^2+z^2}} \right). \tag{18}$$

Adding these terms together yields

$$\boldsymbol{\nabla} \cdot \boldsymbol{V} = \frac{1}{x^2+y^2+z^2}\left(3\sqrt{x^2+y^2+z^2} - \frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}} \right)$$

$$= \frac{1}{x^2+y^2+z^2}\left(3\sqrt{x^2+y^2+z^2} - \sqrt{x^2+y^2+z^2} \right)$$

$$= \frac{2}{\sqrt{x^2+y^2+z^2}}\,. \tag{19}$$

In spherical polar coordinates, this reduces to

$$\boldsymbol{\nabla}\cdot\boldsymbol{V}=\frac{2}{r}\,, \tag{20}$$

and the volume integral of this divergence is

$$\iiint\boldsymbol{\nabla}\cdot\boldsymbol{V}\,d\tau=2\underbrace{\int_0^1 r\,dr}_{\frac{1}{2}}\underbrace{\int_0^{2\pi}d\phi}_{2\pi}\underbrace{\int_0^{\pi}\sin\theta\,d\theta}_{2}=4\pi\,, \tag{21}$$

in agreement with Eq. (14).

3. The gradient of the scalar function $\Phi(r)$, where $r=(x^2+y^2)^{1/2}$, is

$$\boldsymbol{\nabla}\Phi=\frac{\partial\Phi}{\partial x}+\frac{\partial\Phi}{\partial y}=\frac{d\Phi}{dr}\frac{\partial r}{\partial x}\,\boldsymbol{i}+\frac{d\Phi}{dr}\frac{\partial r}{\partial y}\,\boldsymbol{j}\,. \tag{22}$$

Since

$$\frac{\partial r}{\partial x}=\frac{1}{2}(x^2+y^2)^{-1/2}2x=\frac{x}{r}\,, \tag{23}$$

$$\frac{\partial r}{\partial y}=\frac{1}{2}(x^2+y^2)^{-1/2}2y=\frac{y}{r}\,, \tag{24}$$

we have

$$\boldsymbol{\nabla}\Phi=\frac{d\Phi}{dr}\left(\frac{x}{r}\,\boldsymbol{i}+\frac{y}{r}\,\boldsymbol{j}\right)\,. \tag{25}$$

On the perimeter of the circle of radius R centered at the origin, this expression is

$$\boldsymbol{\nabla}\Phi\bigg|_{r=R}=\frac{d\Phi}{dr}\bigg|_{r=R}\left(\frac{x}{R}\,\boldsymbol{i}+\frac{y}{R}\,\boldsymbol{j}\right)\,. \tag{26}$$

The outward unit normal along the perimeter of the circle is determined by first taking the gradient of x^2+y^2,

$$\boldsymbol{\nabla}(x^2+y^2)=2x\,\boldsymbol{i}+2y\,\boldsymbol{j}\,, \tag{27}$$

and normalizing, to obtain

$$\boldsymbol{n}=\frac{x}{r}\,\boldsymbol{i}+\frac{y}{r}\,\boldsymbol{j}\,, \tag{28}$$

which, over the perimeter of the circle is

$$\boldsymbol{n}=\frac{x}{R}\,\boldsymbol{i}+\frac{y}{R}\,\boldsymbol{j}\,. \tag{29}$$

Thus,

$$\boldsymbol{\nabla} V \cdot \boldsymbol{n} = \left.\frac{d\Phi}{dr}\right|_{r=R} \left(\frac{x^2+y^2}{R^2}\right) = \left.\frac{d\Phi}{dr}\right|_{r=R}, \tag{30}$$

so

$$\int \boldsymbol{\nabla} V \cdot \boldsymbol{n}\, d\sigma = \int_0^{2\pi} \left.\frac{d\Phi}{dr}\right|_{r=R} R\, d\phi = 2\pi R \left.\frac{d\Phi}{dr}\right|_{r=R}. \tag{31}$$

For the right-hand side to be independent of R, we must have that

$$\phi(r) = A \ln r\,, \tag{32}$$

where A is an constant. In this case, we obtain

$$\int \boldsymbol{\nabla} V \cdot \boldsymbol{n}\, d\sigma = 2\pi A\,. \tag{33}$$

The same discussion in the course notes for the three-dimensional case can now be applied here to obtain Gauss's law in two dimensions. The main difference is that the Coulomb potential is replaced by $\phi(r) = A \ln r$.