First-Year Mathematics

Problem Set 8 February 25, 2005

The divergence theorem in three dimensions is

$$\iiint \boldsymbol{\nabla}\cdot\boldsymbol{V}\,d\tau = \iint \boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma$$

for a volume τ bounded by a surface σ.

1. Evaluate both sides of this equation for

$$\boldsymbol{V} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + xyz\,\boldsymbol{k}\,,$$

where the volume τ is the region $x^2 + y^2 + z^2 \le 1$. Proceed by following the steps outlined below.

(a) To evaluate the right-hand side of the divergence theorem, begin by showing that the outward unit normal $\boldsymbol{n}$ to the surface σ surrounding the region τ is

$$\boldsymbol{n} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}\,.$$

(b) Form the "dot" product $\boldsymbol{V}\cdot\boldsymbol{n}$, use spherical polar coordinates to set up the surface integral on the right-hand side as

$$\iint \boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma = \int_0^{2\pi} d\,\phi \int_0^{\pi} \sin\theta\,d\theta(\sin^2\theta + \cos\phi\sin\phi\sin^2\theta\cos^2\theta)\,,$$

and show that this integral evaluates to $\frac{8}{3}\pi$.

(c) Calculate divergence of $\boldsymbol{V}$, set up the volume integral in spherical polar coordinates,

$$\iiint \boldsymbol{\nabla}\cdot\boldsymbol{V}\,d\tau = \int_0^1 r^2\,dr \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta(2 + r^2\cos\phi\sin\phi\sin^2\theta)\,,$$

evaluate this integral, and thereby verify the divergence theorem.

2. Verify the divergence theorem for

$$\boldsymbol{V} = \frac{x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}}{\sqrt{x^2 + y^2 + z^2}}$$

where τ again is the region $x^2 + y^2 + z^2 \le 1$.

(a) To evaluate the right-hand side of the theorem, show that

$$\boldsymbol{V} \cdot \boldsymbol{n} = \sqrt{x^2 + y^2 + z^2}$$

and evaluate the surface integral to obtain

$$\iint \boldsymbol{V} \cdot \boldsymbol{n} \, d\sigma = 4\pi \, .$$

(b) For the right-hand side of the divergence theorem, show that

$$\boldsymbol{\nabla} \cdot \boldsymbol{V} = \frac{2}{\sqrt{x^2 + y^2 + z^2}}$$

and evaluate the integral over τ to obtain the same result as in (a).

3. Consider Gauss' law in two dimensions. Let $\boldsymbol{V}$ be defined by

$$\boldsymbol{V} = \boldsymbol{\nabla}\Phi(r) \, ,$$

where $r = (x^2 + y^2)^{1/2}$. By following analogous steps to those in Sec. 6.3 of the course notes, show that on the perimeter of a circle of radius R centered at the origin,

$$\boldsymbol{V} \cdot \boldsymbol{n} = \left.\frac{d\Phi}{dr}\right|_{r=R} .$$

Hence, by integrating the quantity $\boldsymbol{V} \cdot \boldsymbol{n}$ over the perimeter of a circle of radius R, obtain

$$\iint \boldsymbol{V} \cdot \boldsymbol{n} \, d\sigma = 2\pi R \left.\frac{d\Phi}{dr}\right|_{r=R} .$$

Determine the function $\Phi(r)$ that ensures that the right-hand side is independent of R. This is the two-dimensional analogue of the Coulomb potential in three dimensions.