First-Year Mathematics

Solutions to Problem Set 7 February 18, 2005

1. The divergence of a vector field $\boldsymbol{V} = P(x,y,z)\,\boldsymbol{i} + Q(x,y,z)\,\boldsymbol{j} + R(x,y,z)\,\boldsymbol{k}$ is

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\,. \tag{1}$$

   (a) $\boldsymbol{V} = y\,\boldsymbol{i} + z\,\boldsymbol{j} + x\,\boldsymbol{k}$

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = \frac{\partial(y)}{\partial x} + \frac{\partial(z)}{\partial y} + \frac{\partial(x)}{\partial z} = 0+0+0 = 0\,. \tag{2}$$

   (b) $\boldsymbol{V} = 2\,\boldsymbol{i} - \boldsymbol{j} + (y-4z)\,\boldsymbol{k}$

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = \frac{\partial(2)}{\partial x} - \frac{\partial(1)}{\partial y} + \frac{\partial(y-4z)}{\partial z} = 0+0-4 = -4\,. \tag{3}$$

   (c) $\boldsymbol{V} = 3x^2y\,\boldsymbol{i} - 2y^2x\,\boldsymbol{j} + xyz\,\boldsymbol{k}$

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = 3\frac{\partial(x^2y)}{\partial x} - 2\frac{\partial(y^2x)}{\partial y} + \frac{\partial(xyz)}{\partial z} = 6xy - 4xy + xy = 3xy\,. \tag{4}$$

2. The gradient of a scalar function $f(x,y,z)$ is

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\,\boldsymbol{i} + \frac{\partial f}{\partial y}\,\boldsymbol{j} + \frac{\partial f}{\partial z}\,\boldsymbol{k}\,. \tag{5}$$

   The gradient is a vector field, so we can compute its divergence by applying Eq. (1) with

$$P = \frac{\partial f}{\partial x}\,, \quad Q = \frac{\partial f}{\partial y}\,, \quad R = \frac{\partial f}{\partial z}\,. \tag{6}$$

   We obtain

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} f) &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\,.\end{aligned} \tag{7}$$

   Alternatively, beginning with the "del" operation

$$\boldsymbol{\nabla} = \boldsymbol{i}\,\frac{\partial}{\partial x} + \boldsymbol{j}\,\frac{\partial}{\partial y} + \boldsymbol{k}\,\frac{\partial}{\partial z}\,, \tag{8}$$

the "dot" product $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \boldsymbol{\nabla}^2$ yields

$$\begin{aligned} \boldsymbol{\nabla}^2 &= \left(\boldsymbol{i}\,\frac{\partial}{\partial x} + \boldsymbol{j}\,\frac{\partial}{\partial y} + \boldsymbol{k}\,\frac{\partial}{\partial z} \right) \cdot \left(\boldsymbol{i}\,\frac{\partial}{\partial x} + \boldsymbol{j}\,\frac{\partial}{\partial y} + \boldsymbol{k}\,\frac{\partial}{\partial z} \right) \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \,. \end{aligned} \tag{9}$$

We thereby obtain

$$\boldsymbol{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \,, \tag{10}$$

which agrees with Eq. (7).

To take the Laplacian of

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{1}{r} \,, \tag{11}$$

we calculate the partial derivatives with respect to x, y,, and z by applying the chain rule:

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) &= \frac{\partial}{\partial x}\left[\left(x^2+y^2+z^2\right)^{-1/2}\right] \\ &= -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2} 2x \\ &= -x\left(x^2+y^2+z^2\right)^{-3/2} \,. \\ \frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) &= \frac{\partial}{\partial x}\left[-x\left(x^2+y^2+z^2\right)^{-3/2}\right] \\ &= -\left(x^2+y^2+z^2\right)^{-3/2} + x\left(-\frac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2} 2x \\ &= -\left(x^2+y^2+z^2\right)^{-3/2} + 3x^2\left(x^2+y^2+z^2\right)^{-5/2} \,. \end{aligned} \tag{12}$$

Essentially identical calculations yield

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = -\left(x^2+y^2+z^2\right)^{-3/2} + 3y^2\left(x^2+y^2+z^2\right)^{-5/2} \tag{13}$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = -\left(x^2+y^2+z^2\right)^{-3/2} + 3z^2\left(x^2+y^2+z^2\right)^{-5/2} \,. \tag{14}$$

The Laplacian of the function in Eq. (11) is obtained by adding together the derivatives calculated in Eqs. (12), (13), and (14):

$$\begin{aligned}
\boldsymbol{\nabla}^2 \left(\frac{1}{\sqrt{x^2+y^2+z^2}} \right) &= -3\left(x^2+y^2+z^2\right)^{-3/2} + 3(x^2+y^2+z^2)\left(x^2+y^2+z^2\right)^{-5/2} \\
&= -3\left(x^2+y^2+z^2\right)^{-3/2} + 3\left(x^2+y^2+z^2\right)^{-3/2} \\
&= 0\,. \qquad (15)
\end{aligned}$$

3. (a) The divergence of $\boldsymbol{V} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$ is

$$\boldsymbol{\nabla}\cdot\mathbf{V} = \frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} = 1 + 1 = 2\,.$$

The integral of this divergence over the interior of a circle of radius R can be done by inspection, or by using polar coordinates:

$$2\underbrace{\int_0^R r\,dr}_{\frac{1}{2}R^2}\underbrace{\int_0^{2\pi} d\phi}_{2\pi} = 2\pi R^2\,.$$

(b) The equation for the circle is $x^2+y^2=R^2$ The gradient of this expression is

$$\boldsymbol{\nabla}(x^2+y^2) = 2x\,\boldsymbol{i} + 2y\,\boldsymbol{j}\,.$$

By taking the “dot” product of this vector with itself and using the fact that x and y lie on a circle of radius R, we obtain

$$(2x\,\boldsymbol{i} + 2y\,\boldsymbol{j})\cdot(2x\,\boldsymbol{i} + 2y\,\boldsymbol{j}) = 4x^2 + 4y^2 = 4R^2\,.$$

The outward unit normal $\boldsymbol{n}$ to the circle is therefore given by

$$\boldsymbol{n} = \frac{x}{R}\,\boldsymbol{i} + \frac{y}{R}\,\boldsymbol{j}\,.$$

Thus,

$$\boldsymbol{V}\cdot\boldsymbol{n} = \frac{x^2}{R} + \frac{y^2}{R} = R\,,$$

so the integral of this quantity over the circumference of the circle is

$$\int \boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma = R\times 2\pi R = 2\pi R^2\,,$$

which agrees with the result obtained in Part (a).

4. (a) The boundary of the semi-circular region is given by the equation $x^2+y^2=1$, so the outward normal is obtained by taking the gradient of this expression:

$$\boldsymbol{\nabla}(x^2+y^2)=2x\,\boldsymbol{i}+2y\,\boldsymbol{j}\,, \tag{16}$$

The corresponding *unit* vector is obtained by dividing this vector by its magnitude on a circle of unit radius,

$$|\boldsymbol{\nabla}(x^2+y^2)|=2\,, \tag{17}$$

to obtain

$$\boldsymbol{n}=\frac{\boldsymbol{\nabla}(x^2+y^2)}{|\boldsymbol{\nabla}(x^2+y^2)|}=x\,\boldsymbol{i}+y\,\boldsymbol{j}\,. \tag{18}$$

For the straight part of the boundary, the unit normal is see by inspection to be $\boldsymbol{n}=-\boldsymbol{j}$. It can also be calculated by taking gradient the equation of this line, $y=\text{constant}$,

$$\boldsymbol{n}=-\boldsymbol{\nabla}x=-\boldsymbol{j}\,, \tag{19}$$

where the minus sign is inserted to make the vector point along the outward direction of the enclosed surface. The integral of $\boldsymbol{V}\cdot\boldsymbol{n}$ over the boundary is therefore given by the sum of two integrals: one over the semi-circular segment, σ_1 and one over the straight segment, σ_2:

$$\begin{aligned}\int \boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma &= \int (xy\,\boldsymbol{i}+x^2\,\boldsymbol{j})\cdot(x\,\boldsymbol{i}+y\,\boldsymbol{j})\,d\sigma_1+\int (xy\,\boldsymbol{i}+x^2\,\boldsymbol{j})\cdot(-\boldsymbol{j})\,d\sigma_2\\ &= 2\int x^2y\,d\sigma_1-\int x^2\,d\sigma_2\,.\end{aligned} \tag{20}$$

Using circular polar coordinates for the first integral, with

$$x=\cos\phi\,,\qquad y=\sin\phi\,,\qquad d\sigma_1=d\phi\,, \tag{21}$$

and $d\sigma_2=dx$ for the second integral, we obtain

$$\begin{aligned}\int \boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma &= 2\int_0^\pi \cos^2\phi\sin\phi\,d\phi-\int_{-1}^1 x^2\,dx\\ &= -\frac{2}{3}\cos^3\phi\Big|_0^\pi-\frac{1}{3}x^3\Big|_{-1}^1\\ &= \frac{4}{3}-\frac{2}{3}=\frac{2}{3}\,.\end{aligned} \tag{22}$$

(b) The divergence of $\boldsymbol{V}$ is

$$\boldsymbol{\nabla}\cdot\boldsymbol{V}=y\,. \tag{23}$$

Thus, using polar coordinates,

$$y=r\sin\phi\,,\qquad d\tau=r\,dr\,d\phi\,, \tag{24}$$

we obtain

$$\iint \boldsymbol{\nabla}\cdot\boldsymbol{V}\,d\tau = \int_0^1 r\,dr\int_0^\pi d\phi(r\sin\phi)$$
$$= \underbrace{\int_0^1 r^2\,dr}_{\frac{1}{3}}\underbrace{\int_0^\pi \sin\phi\,d\phi}_{2} = \frac{2}{3}\,, \tag{25}$$

which is the same result as in Eq. (22).

5. Upon dividing the interval (a, b) divided into N subintervals of length $\Delta x_N = (b-a)/N$, we have

$$\sum_{n=0}^{N-1}\left[f(x+\Delta x)-f(x)\right]\Big|_{x=a+n\Delta x+N} = [f(a+\Delta x)-f(a)]$$
$$+[f(a+2\Delta x)-f(x+\Delta x)]+\cdots+\Big[\underbrace{f(x+N\Delta x_N)}_{f(b)}-f(x+(N-1)\Delta x_N)\Big]$$
$$= f(b)-f(a)\,, \tag{26}$$

because of cancellation on neighboring intervals. Thus,

$$\lim_{N\to\infty}\left[\sum_{n=0}^{N-1}\frac{df}{dx}\bigg|_{x=a+n\Delta x_N}\Delta x_N\right] = \int_a^b \frac{df}{dx}\,dx = f(b)-f(a)\,. \tag{27}$$

The Fundamental Theorem of Calculus is

$$\int_a^b f(x)\,dx = F(b)-F(a)\,,\qquad \frac{dF}{dx} = f\,, \tag{28}$$

which we can write as

$$\int_a^b \frac{dF}{dx}(x)\,dx = F(b)-F(a)\,. \tag{29}$$

By comparing Eqs. (27) with (29), we conclude that the two are equivalent.

6. The equation of the circle is

$$(x-x_0)^2+(y-y_0)^2 = R^2\,, \tag{30}$$

which can be parametrized in circular polar coordinates as

$$x = x_0 + R\cos\phi\,,\qquad y = y_0 + R\sin\phi\,, \tag{31}$$

where $0 \le \phi < 2\pi$. To calculate the flux $\boldsymbol{V}\cdot\boldsymbol{n}$ through the circular boundary, we first determine the outward normal $\boldsymbol{n}$ at the boundary by using the gradient:

$$\boldsymbol{\nabla}\left[(x-x_0)^2+(y-y_0)^2\right] = 2(x-x_0)\,\boldsymbol{i}+2(y-y_0)\,\boldsymbol{j}\,. \tag{32}$$

Along the circular boundary, x and y are given by Eq. (31), so this expression reduces to

$$\boldsymbol{\nabla}\left[(x-x_0)^2+(y-y_0)^2\right] = 2\cos\phi\,\boldsymbol{i}+2\sin\phi\,\boldsymbol{j}\,. \tag{33}$$

The corresponding unit vector is obtained by dividing this vector by its length, which is

$$\left|\boldsymbol{\nabla}\left[(x-x_0)^2+(y-y_0)^2\right]\right| = \left(4\cos^2\phi+4\sin^2\right)^{1/2} = 2\,. \tag{34}$$

Thus,

$$\boldsymbol{n} = \cos\phi\,\boldsymbol{i}+\sin\phi\,\boldsymbol{j}\,, \tag{35}$$

so, on the circle,

$$\begin{aligned}\boldsymbol{V}\cdot\boldsymbol{n} = {} & P(x_0+R\cos\phi, y_0+R\sin\phi)\cos\phi \\ & +Q(x_0+R\cos\phi, y_0+R\sin\phi)\sin\phi\,.\end{aligned} \tag{36}$$

The flux of $\boldsymbol{V}$ through the circular boundary is therefore given by

$$\begin{aligned}F = {} & \int_0^{2\pi}\Big[P(x_0+R\cos\phi, y_0+R\sin\phi)\cos\phi \\ & +Q(x_0+R\cos\phi, y_0+R\sin\phi)\sin\phi\Big]\,R\,d\phi\,.\end{aligned} \tag{37}$$

We can now expand P and Q in Taylor series about the center of the circle. This is essentially an expansion in powers of R, so we keep only the first-order term, since the higher-order terms will vanish in the limit that $R\to 0$. We obtain

$$\begin{aligned}& P(x_0+R\cos\phi, y_0+R\sin\phi) \\ & \quad = P(x_0,y_0)+\frac{\partial P}{\partial x}\bigg|_{x_0,y_0}R\cos\phi+\frac{\partial P}{\partial y}\bigg|_{x_0,y_0}R\sin\phi+\cdots\,,\end{aligned} \tag{38}$$

$$\begin{aligned}& Q(x_0+R\cos\phi, y_0+R\sin\phi) \\ & \quad = Q(x_0,y_0)+\frac{\partial Q}{\partial x}\bigg|_{x_0,y_0}R\cos\phi+\frac{\partial Q}{\partial y}\bigg|_{x_0,y_0}R\sin\phi+\cdots\,.\end{aligned} \tag{39}$$

Substitution of these expansions into Eq. (37) yields

$$\begin{aligned}F = {} & P(x_0,y_0)R\underbrace{\int_0^{2\pi}\cos\phi\,d\phi}_{=0}+Q(x_0,y_0)R\underbrace{\int_0^{2\pi}\sin\phi\,d\phi}_{=0} \\ & +\frac{\partial P}{\partial x}\bigg|_{x_0,y_0}R^2\underbrace{\int_0^{2\pi}\cos^2\phi\,d\phi}_{=\pi}+\frac{\partial P}{\partial y}\bigg|_{x_0,y_0}R^2\underbrace{\int_0^{2\pi}\sin\phi\cos\phi\,d\phi}_{=0}+\cdots \\ & +\frac{\partial Q}{\partial x}\bigg|_{x_0,y_0}R^2\underbrace{\int_0^{2\pi}\sin\phi\cos\phi\,d\phi}_{=0}+\frac{\partial P}{\partial y}\bigg|_{x_0,y_0}R^2\underbrace{\int_0^{2\pi}\sin^2\phi\,d\phi}_{=\pi}+\cdots\end{aligned}$$

$$= \pi R^2 \left(\left.\frac{\partial P}{\partial x}\right|_{x_0,y_0} + \left.\frac{\partial Q}{\partial y}\right|_{x_0,y_0} \right) + \cdots \tag{40}$$

Dividing both sides of this equation by the area $A = \pi R^2$ of the circle and taking the limit that $R \to 0$ yields

$$\lim_{A\to 0} \left(\frac{F}{A} \right) = \left. \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \right|_{x_0,y_0}, \tag{41}$$

which is the divergence of $\boldsymbol{V}$ at (x_0, y_0).