# First-Year Mathematics

Problem Set 7 February 18, 2005

1. Find the divergence of each of the following vector fields:

   (a) $\boldsymbol{V} = y\,\boldsymbol{i} + z\,\boldsymbol{j} + x\,\boldsymbol{k}$

   (b) $\boldsymbol{V} = 2\,\boldsymbol{i} - \boldsymbol{j} + (y-4z)\,\boldsymbol{k}$

   (c) $\boldsymbol{V} = 3x^2y\,\boldsymbol{i} - 2y^2x\,\boldsymbol{j} + xyz\,\boldsymbol{k}$

2. Write down the gradient of a function $f(x,y,z)$. Take the divergence of this gradient, i.e. compute

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} f)\,.$$

   Show that you obtain the same result by calculating

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})f\,.$$

   This operation, which is written as $\boldsymbol{\nabla}^2 f$, is quite common in physics and is called the **Laplacian** of $f$.

   Suppose that

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{1}{r}\,,$$

   where $r$ is the distance to the origin. Show that

$$\boldsymbol{\nabla}^2\left(\frac{1}{r}\right) = 0\,.$$

   This result is of fundamental importance in electrostatics.

3. The divergence theorem in two dimensions is

$$\iint \boldsymbol{\nabla}\cdot\boldsymbol{V}\,d\tau = \int \boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma$$

   for an area $\tau$ bounded by a closed curve $\sigma$.

   (a) Consider the vector field

$$\boldsymbol{V} = x\,\boldsymbol{i} + y\,\boldsymbol{j}\,.$$

   Find the divergence of $\boldsymbol{V}$ and integrate this quantity over the interior of a circle of radius $R$.

   Answer: $2\pi R^2$.

(b) Evaluate the flux of $\boldsymbol{V}$ by following the steps below:

i. You first need to determine the outward unit normal $\boldsymbol{n}$ to the circle. This is most easily done by taking the gradient of the equation of the circle (in rectangular coordinates) and normalizing.

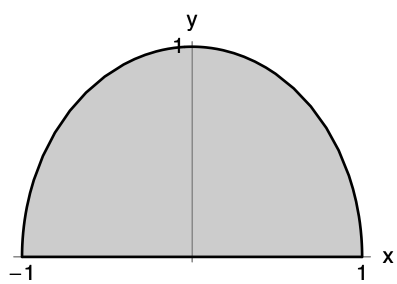
Answer: $\boldsymbol{n} = (x/R)\,\boldsymbol{i} + (y/R)\,\boldsymbol{j}$.

ii. Take the "dot" product $\boldsymbol{V}\cdot\boldsymbol{n}$, integrate this quantity over the circumference of the circle, and show that the result is the same as that obtained in Part 4.

4. Evaluate both sides of the divergence theorem for

$$\boldsymbol{V} = xy\,\boldsymbol{i} + x^2\,\boldsymbol{j}\,,$$

where the area $\tau$ is the region $x^2 + y^2 \le 1$ with $y \ge 0$, as shown below:




Proceed as follows:

(a) Determine the outward unit normal for the semicircular boundary and for the straight boundary between $x = -1$ and $x = 1$. Using polar coordinates for the circular part of the boundary, show that the right-hand side of the divergence theorem is

$$\int \boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma = 2\int_0^\pi \cos^2\phi\sin\phi\,d\phi - \int_{-1}^1 x^2\,dx = \tfrac{2}{3}\,.$$

(b) Calculate the divergence of $\boldsymbol{V}$ and, using polar coordinates, show that the left-hand side of the divergence theorem is

$$\iint \boldsymbol{\nabla}\cdot\boldsymbol{V}\,d\tau = \int_0^1 r\,dr\int_0^\pi d\phi(r\sin\phi)\,,$$

and thus obtain the same result as in (a).

5. Our method for deriving the divergence theorem can be applied to deriving the fundamental theorem of calculus. The definition of the derivative of a function $f(x)$ is

$$\frac{df}{dx} = \lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right].$$

Consider an interval $(a,b)$ divided into $N$ subintervals of length $\Delta x_N = (b-a)/N$ and form the quantity

$$\sum_{n=0}^{N-1} \frac{df}{dx}\bigg|_{x=a+n\Delta x_N} \Delta x_N\,,$$

which is $df/dx$ evaluated at the first point of each subinterval multiplied by the length of the subinterval, i.e. a *Riemann sum*. The definition of the derivative allows us to approximate this sum by

$$\sum_{n=0}^{N-1}\Big[f(x+\Delta x)-f(x)\Big]\bigg|_{x=a+n\Delta x_N}.$$

Show that this quantity is equal to $f(b)-f(a)$. In the limit $N\to\infty$, the Riemann sum becomes an integral, yielding

$$\lim_{N\to\infty}\left[\sum_{n=1}^{N}\frac{df}{dx}\bigg|_{x=a+n\Delta x_N}\Delta x_N\right] = \int_a^b \frac{df}{dx}\,dx = f(b)-f(a)\,.$$

Show that this is the Fundamental Theorem of Calculus.

6.* An alternative to the method used in lectures for deriving the divergence is to consider the flux of a vector field $\boldsymbol{V}(x,y) = P(x,y)\,\boldsymbol{i} + Q(x,y)\,\boldsymbol{j}$ through a circle of radius $R$ centered at $(x_0,y_0)$. The boundary of this circle is given by the set of points

$$x = x_0 + R\cos\phi\,,\quad y = y_0 + R\sin\phi\,,$$

where $0\le\phi<2\pi$. To calculate the flux $\boldsymbol{V}\cdot\boldsymbol{n}$ through the boundary, use the gradient to show that

$$\boldsymbol{n} = \frac{x-x_0}{R}\,\boldsymbol{i} + \frac{y-y_0}{R}\,\boldsymbol{j} = \cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j}\,.$$

Hence, show that the flux $F$ of $\boldsymbol{V}$ through the circle is

$$F = \int_0^{2\pi}\Big[P(x_0+R\cos\phi, y_0+R\sin\phi)\cos\phi$$

$$+Q(x_0+R\cos\phi, y_0+R\sin\phi)\sin\phi\Big]\,R\,d\phi\,.$$

As $R\to 0$, it is sufficient to consider only the first-order terms in the Taylor expansions of $P$ and $Q$ about $(x_0,y_0)$, e.g.

$$P(x_0+R\cos\phi, y_0+R\sin\phi)$$

$$= P(x_0,y_0) + \frac{\partial P}{\partial x}\bigg|_{x_0,y_0} R\cos\phi + \frac{\partial P}{\partial y}\bigg|_{x_0,y_0} R\sin\phi + \cdots\,.$$

Thus, show that the flux is

$$F = \pi R^2 \left( \left.\frac{\partial P}{\partial x}\right|_{x_0,y_0} + \left.\frac{\partial Q}{\partial y}\right|_{x_0,y_0} + \cdots \right).$$

Dividing both sides of this equation by the area $A = \pi R^2$ of the circle and taking the limit $R \to 0$ yields

$$\lim_{A\to 0} \left(\frac{F}{A}\right) = \left.\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\right|_{x_0,y_0},$$

which is the divergence of $\boldsymbol{V}$ at $(x_0, y_0)$.