First-Year Mathematics

Solutions to Problem Set 6 — February 11, 2005

1. The gradient of a function $f(x, y, z)$ is

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\,\boldsymbol{i} + \frac{\partial f}{\partial y}\,\boldsymbol{j} + \frac{\partial f}{\partial z}\,\boldsymbol{k}\,, \tag{1}$$

where, for a function $f(x, y)$ of two variables, the last term is absent. The gradient at a point $\boldsymbol{r}_0 = x_0\,\boldsymbol{i} + y_0\,\boldsymbol{j} + z_0\boldsymbol{k}$ is obtained by evaluating each of the partial derivatives at that point:

$$\boldsymbol{\nabla} f\Big|_{\boldsymbol{r}_0} = \frac{\partial f}{\partial x}\,\boldsymbol{i}\Big|_{\boldsymbol{r}_0} + \frac{\partial f}{\partial y}\,\boldsymbol{j}\Big|_{\boldsymbol{r}_0} + \frac{\partial f}{\partial z}\,\boldsymbol{k}\Big|_{\boldsymbol{r}_0}\,. \tag{2}$$

(a) $f(x, y) = x^2 - y^2$ at $(1, 2)$. The gradient of f is

$$\boldsymbol{\nabla} f = 2x\,\boldsymbol{i} - 2y\,\boldsymbol{j}\,. \tag{3}$$

At $(1, 2)$,

$$\boxed{\boldsymbol{\nabla}(x^2 - y^2)\Big|_{(1,2)} = 2\,\boldsymbol{i} - 4\,\boldsymbol{j}\,.} \tag{4}$$

(b) $f(x, y, z) = xy + yz + xz$ at $(-1, -1, 0)$. The gradient of f

$$\boldsymbol{\nabla} f = (y + z)\,\boldsymbol{i} + (x + z)\,\boldsymbol{j} + (x + y)\,\boldsymbol{k}\,. \tag{5}$$

At $(-1, -1, 0)$,

$$\boxed{\boldsymbol{\nabla}(xy + yz + xz)\Big|_{(-1,-1,0)} = -\,\boldsymbol{i} - \,\boldsymbol{j} - 2\,\boldsymbol{k}\,.} \tag{6}$$

(c) $f(x, y, z) = e^x \cos(yz)$ at $(1, 0, 1)$. The gradient of f is

$$\boldsymbol{\nabla} f = e^x \cos(yz)\,\boldsymbol{i} - ze^x \sin(yz)\,\boldsymbol{i} - ye^x \sin(yz)\,\boldsymbol{i}\,. \tag{7}$$

At $(1, 0, 1)$.

$$\boxed{\boldsymbol{\nabla}\left[e^x \cos(yz)\right]\Big|_{(1,0,1)} = e^x\,\boldsymbol{i}\,.} \tag{8}$$

2. The directional derivative of f along the direction of the *unit vector* $\boldsymbol{u}$ at the point $\boldsymbol{r}_0 = x_0\,\boldsymbol{i} + y_0\,\boldsymbol{j} + z_0\boldsymbol{k}$ is

$$\frac{df}{ds} = (\boldsymbol{\nabla} f \cdot \boldsymbol{u})\Big|_{\boldsymbol{r}_0}. \tag{9}$$

(a) $f(x,y) = \sin x \sin y$ along $\boldsymbol{i} + \boldsymbol{j}$ at $(0, \frac{1}{4}\pi)$. The gradient of f is

$$\boldsymbol{\nabla} f = \cos x \sin y\,\boldsymbol{i} + \sin x \cos y\,\boldsymbol{j}\,. \tag{10}$$

The length of the given vector $|\boldsymbol{i} + \boldsymbol{j}| = \sqrt{2}$, so the corresponding *unit* vector is

$$\boldsymbol{u} = \frac{1}{\sqrt{2}}(\boldsymbol{i} + \boldsymbol{j}) = \tfrac{1}{2}\sqrt{2}\,(\boldsymbol{i} + \boldsymbol{j})\,. \tag{11}$$

The directional derivative is thus given by

$$\boldsymbol{\nabla} f \cdot \boldsymbol{u} = \tfrac{1}{2}\sqrt{2}(\cos x \sin y + \sin x \cos y)\,. \tag{12}$$

At $(0, \frac{1}{4}\pi)$,

$$\boxed{(\boldsymbol{\nabla} f \cdot \boldsymbol{u})\Big|_{(0,\frac{1}{4}\pi)} = \tfrac{1}{2}\,.} \tag{13}$$

(b) $f(x,y) = e^{-x^2-y^2}$ along $\boldsymbol{i}$ at $(0,1)$. The gradient of f is

$$\boldsymbol{\nabla} f = -2e^{-x^2-y^2}(x\,\boldsymbol{i} + y\,\boldsymbol{j})\,. \tag{14}$$

The vector $\boldsymbol{i}$ is already a unit vector, so the directional derivative is

$$\boldsymbol{\nabla} f \cdot \boldsymbol{u} = -2xe^{-x^2-y^2}\,. \tag{15}$$

At $(0,1)$,

$$\boxed{(\boldsymbol{\nabla} f \cdot \boldsymbol{u})\Big|_{(0,1)} = 0\,.} \tag{16}$$

(c) $f(x,y,z) = x^2 + y^2 - z^2$ along $-\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ at $(1,1,1)$. The gradient of f is

$$\boldsymbol{\nabla} f = 2x\,\boldsymbol{i} + 2y\,\boldsymbol{j} - 2z\boldsymbol{k}\,. \tag{17}$$

The length of the vector $|-\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}| = \sqrt{3}$, so the corresponding *unit* vector is

$$\boldsymbol{u} = \frac{1}{\sqrt{3}}(-\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = \tfrac{1}{3}\sqrt{3}(-\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})\,, \tag{18}$$

and the directional derivative is

$$\boldsymbol{\nabla} f \cdot \boldsymbol{u} = -\tfrac{2}{3}\sqrt{3}(x+y+z)\,. \tag{19}$$

At $(1,1,1)$,

$$\boxed{(\boldsymbol{\nabla} f \cdot \boldsymbol{u})\Big|_{(1,1,1)} = -2\sqrt{3}\,.} \tag{20}$$

3. The gradient of $f(x,y,z) = ax + by + cz$ is

$$\boldsymbol{\nabla} f = a\,\boldsymbol{i} + b\,\boldsymbol{j} + c\,\boldsymbol{k}\,. \tag{21}$$

Since the gradient is normal to the surfaces of constant f, this vector must be normal to

$$ax + by + cz = d\,, \tag{22}$$

which is the equation of a plane.

4. The equation of the tangent plane to a surface $f(x,y,z) =$ constant at a particular point is determined as follows.

(a) The gradient of f at $\boldsymbol{r}_0$ is

$$\boldsymbol{\nabla} f\Big|_{\boldsymbol{r}_0} = \frac{\partial f}{\partial x}\Big|_{\boldsymbol{r}_0}\boldsymbol{i} + \frac{\partial f}{\partial y}\Big|_{\boldsymbol{r}_0}\boldsymbol{j} + \frac{\partial f}{\partial z}\Big|_{\boldsymbol{r}_0}\boldsymbol{k}\,. \tag{23}$$

(b) The tangent plane to the surface $f(x,y,z) =$ constant at $\boldsymbol{r}_0$ has the general form

$$ax + by + cz = d\,, \tag{24}$$

where a, b, c, and d are constants. The normal vector $\boldsymbol{n}$ to this plane is, from Part 3, given by

$$\boldsymbol{n} = a\,\boldsymbol{i} + b\,\boldsymbol{j} + c\,\boldsymbol{k}\,. \tag{25}$$

Since this vector is parallel to the gradient at $\boldsymbol{r}_0$, we can equate the components of $\boldsymbol{n}$ and $\boldsymbol{\nabla} f$ to obtain

$$a = \frac{\partial f}{\partial x}\Big|_{\boldsymbol{r}_0}, \qquad b = \frac{\partial f}{\partial y}\Big|_{\boldsymbol{r}_0}, \qquad c = \frac{\partial f}{\partial z}\Big|_{\boldsymbol{r}_0}. \tag{26}$$

(c) The tangent plane must pass through the point $\boldsymbol{r}_0$. Thus, we must have that

$$d = ax_0 + by_0 + cz_0 = x_0 \frac{\partial f}{\partial x}\bigg|_{\boldsymbol{r}_0} + y_0 \frac{\partial f}{\partial y}\bigg|_{\boldsymbol{r}_0} + z_0 \frac{\partial f}{\partial z}\bigg|_{\boldsymbol{r}_0} . \tag{27}$$

By combining Eqs. (24), (26), (27), the equation for the tangent plane is obtained as

$$\boxed{(x - x_0)\frac{\partial f}{\partial x}\bigg|_{\boldsymbol{r}_0} + (y - y_0)\frac{\partial f}{\partial y}\bigg|_{\boldsymbol{r}_0} + (z - z_0)\frac{\partial f}{\partial z}\bigg|_{\boldsymbol{r}_0} = 0\,.} \tag{28}$$

5. (a) The tangent plane to $x^2 + y^2 + z^2 = 1$ at $(0, 0, 1)$. The three partial derivatives evaluated at $(0, 0, 1)$ are

$$\frac{\partial f}{\partial x}\bigg|_{(0,0,1)} = 2x\bigg|_{(0,0,1)} = 0 \tag{29}$$

$$\frac{\partial f}{\partial y}\bigg|_{(0,0,1)} = 2y\bigg|_{(0,0,1)} = 0 \tag{30}$$

$$\frac{\partial f}{\partial z}\bigg|_{(0,0,1)} = 2z\bigg|_{(0,0,1)} = 2\,. \tag{31}$$

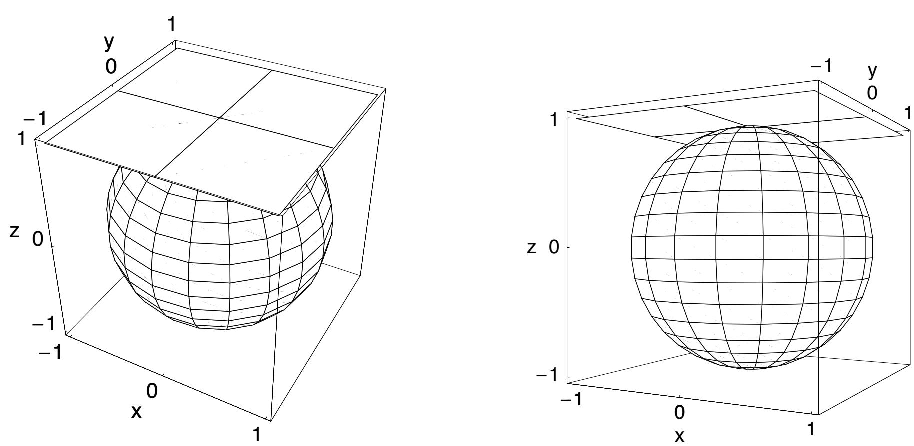
Thus, from Eq. (28), the equation of the tangent plane is

$$2(z - 1) = 0\,, \tag{32}$$

or, simply,

$$\boxed{z = 1\,.} \tag{33}$$

Two views of the surface and the tangent plane are shown below:



(b) The tangent plane to $x^2+xy^2+yz=1$ at $(-1,2,2)$. The three partial derivatives evaluated at $(-1,2,2)$ are

$$\left.\frac{\partial f}{\partial x}\right|_{(-1,2,2)} = \left.(2x+y^2)\right|_{(-1,2,2)} = 2 \tag{34}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(-1,2,2)} = \left.(2xy+z)\right|_{(-1,2,2)} = -2 \tag{35}$$

$$\left.\frac{\partial f}{\partial z}\right|_{(-1,2,2)} = \left.y\right|_{(-1,2,2)} = 2\,. \tag{36}$$

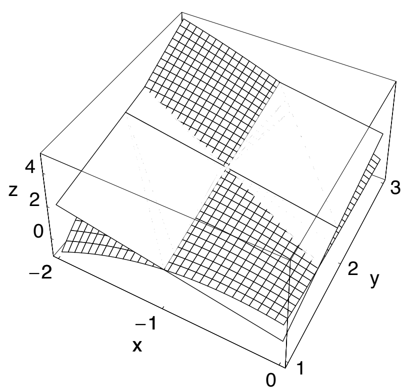
Thus, from Eq. (28), the equation of the tangent plane is

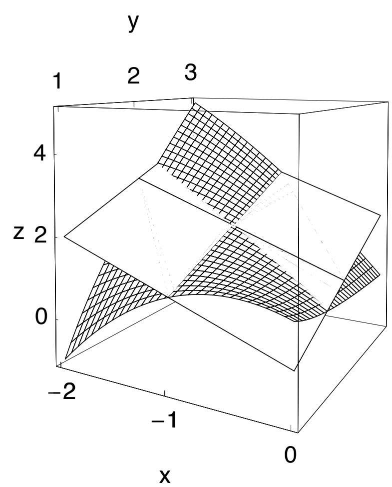
$$2(x+1)-2(y-2)+2(z-2)=0\,, \tag{37}$$

or,

$$\boxed{x-y+z=-1\,.} \tag{38}$$

Two views of the surface and the tangent plane are shown below:





(c) The tangent plane to $z=x^2+y^2$ at $(1,1,2)$. We first write this surface as $x^2+y^2-z=0$. The three partial derivatives evaluated at $(1,1,2)$ are then

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1,2)} = \left.2x\right|_{(1,1,2)} = 2 \tag{39}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,1,2)} = \left.2y\right|_{(1,1,2)} = 2 \tag{40}$$

$$\left.\frac{\partial f}{\partial z}\right|_{(1,1,2)} = \left.-1\right|_{(1,1,2)} = -1\,. \tag{41}$$

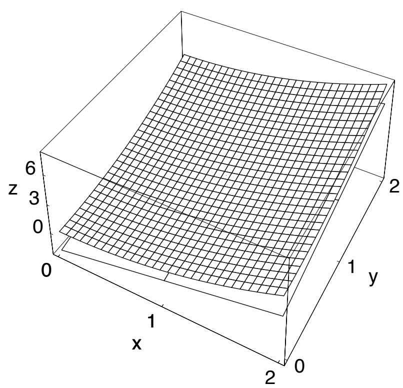
Thus, from Eq. (28), the equation of the tangent plane is

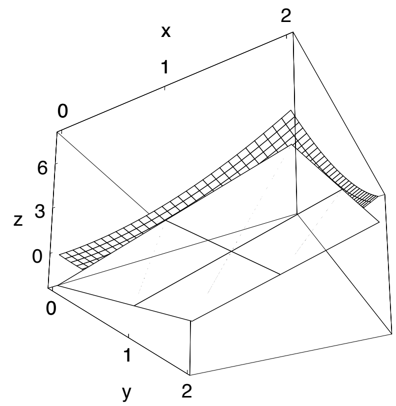
$$2(x-1)+2(y-1)-(z-2)=0\,, \tag{42}$$

or,

$$2x+2y-z=2\,. \tag{43}$$

Two views of the surface and the tangent plane are shown below:





6. All of the following identities are consequences of the definition of the gradient and the properties of partial derivatives.

   (a) $\boldsymbol{\nabla}(af+bg)=a\boldsymbol{\nabla}f+b\boldsymbol{\nabla}g$, where a and b are constants.

$$\begin{aligned}\boldsymbol{\nabla}(af+bg) &= \frac{\partial(af+bg)}{\partial x}\,\boldsymbol{i}+\frac{\partial(af+bg)}{\partial y}\,\boldsymbol{j}+\frac{\partial(af+bg)}{\partial z}\,\boldsymbol{k}\\ &= \frac{\partial(af)}{\partial x}\,\boldsymbol{i}+\frac{\partial(af)}{\partial y}\,\boldsymbol{j}+\frac{\partial(af)}{\partial z}\,\boldsymbol{k}+\frac{\partial(bg)}{\partial x}\,\boldsymbol{i}+\frac{\partial(bg)}{\partial y}\,\boldsymbol{j}+\frac{\partial(bg)}{\partial z}\,\boldsymbol{k}\\ &= a\left(\frac{\partial f}{\partial x}\,\boldsymbol{i}+\frac{\partial f}{\partial y}\,\boldsymbol{j}+\frac{\partial f}{\partial z}\,\boldsymbol{k}\right)+b\left(\frac{\partial g}{\partial x}\,\boldsymbol{i}+\frac{\partial g}{\partial y}\,\boldsymbol{j}+\frac{\partial g}{\partial z}\,\boldsymbol{k}\right)\\ &= a\boldsymbol{\nabla}f+b\boldsymbol{\nabla}g\,. \end{aligned} \tag{44}$$

   (b) $\boldsymbol{\nabla}(fg)=g\boldsymbol{\nabla}f+f\boldsymbol{\nabla}g$.

$$\begin{aligned}\boldsymbol{\nabla}(fg) &= \frac{\partial(fg)}{\partial x}\,\boldsymbol{i}+\frac{\partial(fg)}{\partial y}\,\boldsymbol{j}+\frac{\partial(fg)}{\partial z}\,\boldsymbol{k}\\ &= \left(g\frac{\partial f}{\partial x}+f\frac{\partial g}{\partial x}\right)\boldsymbol{i}+\left(g\frac{\partial f}{\partial y}+f\frac{\partial g}{\partial y}\right)\boldsymbol{j}+\left(g\frac{\partial f}{\partial z}+f\frac{\partial g}{\partial z}\right)\boldsymbol{k}\end{aligned}$$

$$= g\left(\frac{\partial f}{\partial x}\,\boldsymbol{i} + \frac{\partial f}{\partial y}\,\boldsymbol{j} + \frac{\partial f}{\partial z}\,\boldsymbol{k}\right) + f\left(\frac{\partial g}{\partial x}\,\boldsymbol{i} + \frac{\partial g}{\partial y}\,\boldsymbol{j} + \frac{\partial g}{\partial z}\,\boldsymbol{k}\right)$$

$$= g\boldsymbol{\nabla} f + f\boldsymbol{\nabla} g\,. \tag{45}$$

(c) $\boldsymbol{\nabla}(f^n) = nf^{n-1}\boldsymbol{\nabla} f$.

$$\boldsymbol{\nabla}(f^n) = \frac{\partial(f^n)}{\partial x}\,\boldsymbol{i} + \frac{\partial(f^n)}{\partial y}\,\boldsymbol{j} + \frac{\partial(f^n)}{\partial z}\,\boldsymbol{k}$$

$$= \left(nf^{n-1}\frac{\partial f}{\partial x}\,\boldsymbol{i} + nf^{n-1}\frac{\partial f}{\partial y}\,\boldsymbol{j} + nf^{n-1}\frac{\partial f}{\partial z}\,\boldsymbol{k}\right)$$

$$= nf^{n-1}\left(\frac{\partial f}{\partial x}\,\boldsymbol{i} + \frac{\partial f}{\partial y}\,\boldsymbol{j} + \frac{\partial f}{\partial z}\,\boldsymbol{k}\right)$$

$$= \boldsymbol{\nabla} f\,. \tag{46}$$

(d) $\boldsymbol{\nabla}\left(\frac{f}{g}\right) = \frac{1}{g^2}(g\boldsymbol{\nabla} f - f\boldsymbol{\nabla} g)$.

$$\boldsymbol{\nabla}\left(\frac{f}{g}\right) = \left[\frac{\partial}{\partial x}\left(\frac{f}{g}\right)\right]\,\boldsymbol{i} + \left[\frac{\partial}{\partial y}\left(\frac{f}{g}\right)\right]\,\boldsymbol{j} + \left[\frac{\partial}{\partial z}\left(\frac{f}{g}\right)\right]\,\boldsymbol{k}$$

$$= \frac{1}{g^2}\left(g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}\right)\,\boldsymbol{i} + \frac{1}{g^2}\left(g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}\right)\,\boldsymbol{j} + \frac{1}{g^2}\left(g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}\right)\,\boldsymbol{k}$$

$$= \frac{1}{g^2}\left(g\frac{\partial f}{\partial x}\,\boldsymbol{i} + g\frac{\partial f}{\partial y}\,\boldsymbol{j} + g\frac{\partial f}{\partial z}\,\boldsymbol{k}\right) - \frac{1}{g^2}\left(f\frac{\partial g}{\partial x}\,\boldsymbol{i} + f\frac{\partial g}{\partial y}\,\boldsymbol{j} + f\frac{\partial g}{\partial z}\,\boldsymbol{k}\right)$$

$$= \frac{1}{g^2}(g\boldsymbol{\nabla} f - f\boldsymbol{\nabla} g)\,. \tag{47}$$

7. For a differentiable scalar function $\phi(x,y,z)$, the line integral of the gradient of ϕ between two points a and b is

$$\int_a^b \boldsymbol{\nabla}\phi \cdot d\boldsymbol{r} = \int_a^b \left(\frac{\partial \phi}{\partial x}\,\boldsymbol{i} + \frac{\partial \phi}{\partial y}\,\boldsymbol{j} + \frac{\partial \phi}{\partial z}\,\boldsymbol{k}\right)\cdot(dx\,\boldsymbol{i} + dy\,\boldsymbol{j} + dz\,\boldsymbol{k})$$

$$= \int_a^b \left(\frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z}\,dz\right) \tag{48}$$

The right-hand side is recognized as the differential of ϕ, so we can write

$$\int_a^b \boldsymbol{\nabla}\phi \cdot d\boldsymbol{r} = \int_{\phi(a)}^{\phi(b)} d\phi = \phi(b) - \phi(a)\,. \tag{49}$$

The geometrical reasoning behind this result stems from the fact that the gradient is normal to the surfaces of constant ϕ. The integrand of the line integral is the projection of the increment of the path onto $\boldsymbol{\nabla}\phi$. Thus, regardless of the path, only the projection of that path between the initial and final values of ϕ determine the value of the line integral.