First-Year Mathematics

Problem Set 6 February 11, 2005

1. Find the gradient of the following functions at the given point:

   (a) $f(x,y) = x^2 - y^2$ at $(1,2)$.

   (b) $f(x,y,z) = xy + yz + xz$ at $(-1,-1,0)$.

   (c) $f(x,y,z) = e^x \cos(yz)$ at $(1,0,1)$.

   Ans: (a) $2\,\boldsymbol{i} - 4\,\boldsymbol{j}$; (b) $-\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$; (c) $e\,\boldsymbol{i}$.

2. Find the derivatives of each of the following functions along the given direction at the given point:

   (a) $f(x,y) = \sin x \sin y$ along $\boldsymbol{i} + \boldsymbol{j}$ at $(0, \frac{1}{4}\pi)$.

   (b) $f(x,y) = e^{-x^2-y^2}$ along $\boldsymbol{i}$ at $(0,1)$.

   (c) $f(x,y,z) = x^2 + y^2 - z^2$ along $-\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ at $(1,1,1)$.

   Ans: (a) $\frac{1}{2}$; (b) 0; (c) $-2\sqrt{3}$.

3. The equation of a plane is

   $$ax + by + cz = d\,,$$

   where a, b, c, and d are constants. Show that the vector

   $$\boldsymbol{n} = a\,\boldsymbol{i} + b\,\boldsymbol{j} + c\,\boldsymbol{k}$$

   is normal to this plane.

4. Use the result of Part 3 to determine the tangent plane to a surface $f(x,y,z) =$ constant at a point $\boldsymbol{r}_0 = (x_0, y_0, z_0)$. Proceed as follows:

   (a) Write down the expression for the gradient of f at $\boldsymbol{r}_0$.

(b) The vector $\boldsymbol{\nabla} f$ is normal to the tangent plane at this point. Hence, deduce that the quantities a, b, and c in the equation of a plane are given by

$$a = \frac{\partial f}{\partial x}\bigg|_{\boldsymbol{r}_0}, \qquad b = \frac{\partial f}{\partial y}\bigg|_{\boldsymbol{r}_0}, \qquad c = \frac{\partial f}{\partial z}\bigg|_{\boldsymbol{r}_0}.$$

(c) The tangent plane must pass through the point $\boldsymbol{r}_0$. Use this to determine d:

$$d = x_0\frac{\partial f}{\partial x}\bigg|_{\boldsymbol{r}_0} + y_0\frac{\partial f}{\partial y}\bigg|_{\boldsymbol{r}_0} + z_0\frac{\partial f}{\partial z}\bigg|_{\boldsymbol{r}_0}.$$

5. For each of the following surfaces, determine the tangent plane at the given point:

(a) $x^2 + y^2 + z^2 = 1$ at $(0, 0, 1)$.

(b) $x^2 + xy^2 + yz = 1$ at $(-1, 2, 2)$.

(c) $z = x^2 + y^2$ at $(1, 1, 2)$.

Ans: (a) $z = 1$; (b) $x - y + z = -1$; (c) $2x + 2y - z = 2$.

6. Use the definition of the gradient to deduce the following properties:

(a) $\boldsymbol{\nabla}(af + bg) = a\boldsymbol{\nabla} f + b\boldsymbol{\nabla} g$, where a and b are constants.

(b) $\boldsymbol{\nabla}(fg) = g\boldsymbol{\nabla} f + f\boldsymbol{\nabla} g$.

(c) $\boldsymbol{\nabla}(f^n) = nf^{n-1}\boldsymbol{\nabla} f$.

(d) $\boldsymbol{\nabla}\left(\frac{f}{g}\right) = \frac{1}{g^2}(g\boldsymbol{\nabla} f - f\boldsymbol{\nabla} g)$.

7.* Given a differentiable scalar function $\phi(x, y, z)$, show that the line integral of the gradient of ϕ between two points a and b is independent of the path:

$$\int_a^b \boldsymbol{\nabla}\phi \cdot d\boldsymbol{r} = \phi(b) - \phi(a),$$

where $\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$. Use the properties of the gradient to provide a geometrical interpretation of this result.