First-Year Mathematics

Solutions to Problem Set 5 February 5, 2005

1. To write the line integral in terms of x alone, we use $y = x^2$ to eliminate the factor of y in the integrand:
$$I = \int_0^1 x^2\,dx = \left(\frac{x^3}{3}\bigg|_0^1\right) = \frac{1}{3}\,. \tag{1}$$
To write the integral in terms of y alone, we have $x = y^{1/2}$. Thus, since $dy = 2x\,dx$,
$$dx = \frac{dy}{2x} = \frac{dy}{2y^{1/2}}\,. \tag{2}$$
The integral becomes
$$I = \frac{1}{2}\int_0^1 \frac{y\,dy}{y^{1/2}} = \frac{1}{2}\int_0^1 y^{1/2}\,dy = \frac{1}{2}\left(\frac{2}{3}y^{3/2}\bigg|_0^2\right) = \frac{1}{3}\,. \tag{3}$$

2. The relation $x = 2y$ along the path enables us to express each term of the line integral as an integral over x or y alone. By choosing to replace y with x in the first term and x with y in the second term, using the fact that $dx = 2\,dy$ and $1 \le y \le 2$, we obtain
$$\begin{aligned} I &= \int_1^2 \left[x\left(\frac{x}{2}\right)\,dx + (2y)^2 y\,dy\right] = \frac{1}{2}\int_2^4 x^2\,dx + 4\int_1^2 y^3\,dy \\ &= \frac{1}{2}\left(\frac{x^3}{3}\bigg|_2^4\right) + 4\left(\frac{y^4}{4}\bigg|_1^2\right) \\ &= \frac{1}{6}(64-8) + (16-1) = \frac{28}{3} + 15 = \frac{73}{3}\,. \end{aligned} \tag{4}$$

3. Along the upper half-circle, we have
$$x = \cos\phi\,, \qquad y = \sin\phi\,, \tag{5}$$
where $0 \le \phi \le \pi$. Then, with
$$dy = \cos\phi\,d\phi\,, \tag{6}$$
the integral can be written as
$$I = \int_{\mathcal{P}} xy^2\,dy = \int_0^\pi \cos\phi\sin^2\phi(\cos\phi)\,d\phi = \int_0^\pi \cos^2\phi\sin^2\phi\,d\phi \tag{7}$$

The integral on the right-hand side can be evaluated as follows:

$$\int_0^{\pi} \cos^2\phi \sin^2\phi \, d\phi = \frac{1}{4}\int_0^{\pi} \sin 2\phi \, d\phi = \frac{1}{8}\int_0^{2\pi} \sin^2 t \, dt = \frac{\pi}{8} \,. \tag{8}$$

Thus,

$$I = \frac{\pi}{8} \,. \tag{9}$$

4. The line integral

$$\int_{\mathcal{P}} [f(x,y)\,dx + g(x,y)\,dy]$$

is path-independent if, and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

If the line integral is path-independent, the quantity $f\,dx + g\,dy$ is said to be an *exact differential*, in which case there is a potential function F such that

$$\frac{\partial F}{\partial x} = f \,, \qquad \frac{\partial F}{\partial y} = g \,,$$

the integration of which yields F.

(a) $f = x$, $g = y$.

$$\frac{\partial f}{\partial y} = 0 \,, \qquad \frac{\partial g}{\partial x} = 0 \,, \tag{10}$$

so the line integral is *path-independent* and the quantity $x\,dx + y\,dy$ is *exact*.

(b) $f = y$, $g = x$.

$$\frac{\partial f}{\partial y} = 1 \,, \qquad \frac{\partial g}{\partial x} = 1 \,, \tag{11}$$

so the line integral is *path-independent* and the quantity $y\,dx + x\,dy$ is *exact*.

(c) $f = y$, $g = -x$.

$$\frac{\partial f}{\partial y} = 1 \,, \qquad \frac{\partial g}{\partial x} = -1 \,, \tag{12}$$

so the line integral is *path-dependent* and the quantity $y\,dx - x\,dy$ is *inexact*.

(d) $f = \dfrac{x}{\sqrt{x^2+y^2}}\,, g = \dfrac{y}{\sqrt{x^2+y^2}}\,.$

$$\frac{\partial f}{\partial y} = -\frac{xy}{(x^2+y^2)^{3/2}}\,, \qquad \frac{\partial g}{\partial x} = -\frac{xy}{(x^2+y^2)^{3/2}}\,, \tag{13}$$

so the line integral is *path-independent* and the quantity

$$\frac{x\,dx}{\sqrt{x^2+y^2}} + \frac{y\,dy}{\sqrt{x^2+y^2}} \tag{14}$$

is *exact.*

(e) $f = x\cos y,\ g = y\sin x.$

$$\frac{\partial f}{\partial y} = -x\sin y\,, \qquad \frac{\partial g}{\partial x} = y\cos x\,, \tag{15}$$

so the line integral is *path-dependent* and the quantity $x\cos y\,dx + y\sin x\,dy$ is *inexact.*

5. The procedure used to determine the potential function can be combined into a single expression for F. Our derivation will also show that that condition

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}\,, \tag{16}$$

is *sufficient* for the path-independence of the associated line integral,

$$\int_{\mathcal{P}} \left[f(x,y)\,dx + g(x,y)\,dy\right]\,. \tag{17}$$

In lectures we showed the *necessity* of this condition, so we can conclude that this condition is *equivalent* to the path-independence of the line integral.

(a) The integral of $\partial F/\partial x = f$ from x to x_0 is

$$\int_{x_0}^{x} \frac{\partial F(s,y)}{\partial s}\,ds = \int_{x_0}^{x} f(s,y)\,ds\,. \tag{18}$$

The variable s in the "x slot" of F and f is a dummy variable of integration. The Fundamental Theorem of Calculus,

$$\int_a^b f(s)\,ds = F(b) - F(a)\,, \qquad f = \frac{dF}{dx}\,, \tag{19}$$

can be used to evaluate the left-hand side of Eq. (18) as

$$\int_{x_0}^{x} \frac{\partial F(s,y)}{\partial s}\,ds = F(x,y) - F(x_0,y)\,. \tag{20}$$

Therefore, Eq. (18) can be written as

$$\boxed{F(x,y) = F(x_0,y) + \int_{x_0}^{x} f(s,y)\,ds\,.} \tag{21}$$

(b) By differentiating Eq. (21) with respect to y, we obtain

$$\frac{\partial F}{\partial y} = \frac{dF(x_0,y)}{dy} + \int_{x_0}^{x} \frac{\partial f(s,y)}{\partial y}\,ds\,. \tag{22}$$

The integral in this expression can be simplified by using Eq. (16) in the form

$$\frac{\partial f(s,y)}{\partial y} = \frac{\partial g(s,y)}{\partial s}\,. \tag{23}$$

We can then evaluate the integral by again invoking the Fundamental Theorem of Calculus in Eq. (19):

$$\int_{x_0}^{x} \frac{\partial f(s,y)}{\partial y}\,ds = \int_{x_0}^{x} \frac{\partial g(s,y)}{\partial s}\,ds = g(x,y) - g(x_0,y)\,. \tag{24}$$

This, Eq. (22) becomes

$$\boxed{\frac{\partial F}{\partial y} = g(x,y) - g(x_0,y) + \frac{dF(x_0,y)}{dy}\,.} \tag{25}$$

(c) By requiring that Eq. (25) to be equal to the second of Eqs. (16),

$$\frac{\partial F}{\partial y} = g(x,y) - g(x_0,y) + \frac{dF(x_0,y)}{dy} = g(x,y)\,, \tag{26}$$

we must have that

$$\frac{dF(x_0,y)}{dy} = g(x_0,y)\,. \tag{27}$$

Integrating this equation with respect to y and invoking the Fundamental Theorem of Calculus in Eq. (19) yields

$$\int_{y_0}^{y} g(x_0,t)\,dt = \int_{y_0}^{y} \frac{dF(x_0,t)}{dt}\,dt = F(x_0,y) - F(x_0,y_0)\,, \tag{28}$$

or,

$$F(x_0,y) = F(x_0,y_0) + \int_{y_0}^{y} g(x_0,t)\,dt \tag{29}$$

By substituting this expression for $F(x_0, y)$ into Eq. (21), we obtain the following expression for the potential function F:

$$\boxed{F(x,y) = F(x_0,y_0) + \int_{x_0}^{x} f(s,y)\,ds + \int_{y_0}^{y} g(x_0,t)\,dt\,.} \tag{30}$$

(d) The differentiation of Eq. (30) is carried out by invoking Eq. (**??**):

$$\frac{d}{dx}\left[\int_a^x f(s)\,ds\right] = f(x) \tag{31}$$

Thus,

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left[\int_{x_0}^{x} f(s,y)\,ds\right] = f(x)\,, \tag{32}$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[\int_{y_0}^{y} g(x_0,t)\,dt\right] = g(x)\,. \tag{33}$$

6. The differentials in (a), (b), and (d) in Part 4 are exact. The application of Eq. (30) to the determine the potentials is as follows:

(a) $f = x$, $g = y$. In the notation of Eq. (30),

$$f(s,y) = s\,, \qquad g(x_0,t) = t\,. \tag{34}$$

Therefore,

$$\begin{aligned} F(x,y) - F(x_0,y_0) &= \int_{x_0}^{x} s\,ds + \int_{y_0}^{y} t\,dt \\ &= \tfrac{1}{2}s^2\Big|_{x_0}^{x} + \tfrac{1}{2}t^2\Big|_{y_0}^{y} \\ &= \tfrac{1}{2}(x^2 - x_0^2) + \tfrac{1}{2}(y^2 - y_0^2) \\ &= \tfrac{1}{2}(x^2 + y^2) - \tfrac{1}{2}(x_0^2 + y_0^2)\,, \end{aligned} \tag{35}$$

so, to within an additive constant,

$$\boxed{F(x,y) = \tfrac{1}{2}(x^2 + y^2)\,.} \tag{36}$$

(b) $f = y$, $g = x$. In the notation of Eq. (30),

$$f(s, y) = y\,, \qquad g(x_0, t) = x_0\,. \tag{37}$$

Therefore,

$$\begin{aligned} F(x, y) \;-\; F(x_0, y_0) &= \int_{x_0}^{x} y\,ds + \int_{y_0}^{y} x_0\,dt \\ &= y(x - x_0) + x_0(y - y_0) \\ &= xy - x_0 y_0\,, \end{aligned} \tag{38}$$

so, to within an additive constant

$$\boxed{F(x, y) = xy\,.} \tag{39}$$

(d) $f = \dfrac{x}{\sqrt{x^2 + y^2}}\,, g = \dfrac{y}{\sqrt{x^2 + y^2}}$. In the notation of Eq. (30),

$$f(s, y) = \frac{s}{\sqrt{s^2 + y^2}}\,, \qquad g(x_0, t) = \frac{t}{\sqrt{x_0^2 + t^2}}\,. \tag{40}$$

Therefore,

$$\begin{aligned} F(x, y) \;-\; F(x_0, y_0) &= \int_{x_0}^{x} \frac{s\,ds}{\sqrt{s^2 + y^2}} + \int_{y_0}^{y} \frac{t\,dt}{\sqrt{x_0^2 + t^2}} \\ &= \sqrt{s^2 + y^2}\,\Big|_{x_0}^{x} + \sqrt{x_0^2 + t^2}\,\Big|_{y_0}^{y} \\ &= \sqrt{x^2 + y^2} - \sqrt{x_0^2 + y^2} + \sqrt{x_0^2 + y^2} - \sqrt{x_0^2 + y_0^2} \\ &= \sqrt{x^2 + y^2} - \sqrt{x_0^2 + y_0^2}\,, \end{aligned} \tag{41}$$

so, to within an additive constant,

$$\boxed{F(x, y) = \sqrt{x^2 + y^2}\,.} \tag{42}$$