First-Year Mathematics

Problem Set 5 February 5, 2005

1. Evaluate the line integral

$$I = \int_{\mathcal{P}} y\,dx$$

along the path $y = x^2$ from $(0,0)$ to $(1,1)$ in two ways: by writing the line integral in terms of (i) x alone, and (ii) in terms of y alone.

2. Evaluate the integral

$$I = \int_{\mathcal{P}} \left(xy\,dx + x^2y\,dy\right)$$

along the path given by $x = 2y$ from $(2,1)$ to $(4,2)$.

3. Evaluate the integral

$$I = \int_{\mathcal{P}} xy^2\,dy$$

over the upper half-circle $x^2 + y^2 = 1$ from $(1,0)$ to $(-1,0)$.

Hint: Use the fact that, along $\mathcal{P}$, $x = \cos\phi$ and $y = \sin\phi$, where $0 \le \phi \le \pi$ to express the line integral in terms of ϕ.

4. For each of the following pairs of functions f and g, determine if the line integral

$$\int_{\mathcal{P}} [f(x,y)\,dx + g(x,y)\,dy]$$

is path-dependent or path-independent.

(a) $f = x$, $g = y$.

(b) $f = y$, $g = x$.

(c) $f = y$, $g = -x$.

(d) $f = \dfrac{x}{\sqrt{x^2+y^2}}$, $g = \dfrac{y}{\sqrt{x^2+y^2}}$.

(e) $f = x\cos y$, $g = y\sin x$.

5.* We showed in lectures that, if

$$\oint_{\mathcal{C}} [f(x,y)\,dx + g(x,y)\,dy] = 0 \tag{1}$$

for any closed path $\mathcal{C}$, then the functions f and g satisfy

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}\,. \tag{2}$$

This is an "integrability" condition that implies the existence of a potential function F such that

$$\frac{\partial F}{\partial x} = f\,, \quad \frac{\partial F}{\partial y} = g\,. \tag{3}$$

We will show here how to use these equations to obtain a formula for F.

(a) Use the Fundamental Theorem of Calculus to integrate $\partial F/\partial x = f$ from some initial point x_0 to x, while holding y fixed. Show that the result is

$$F(x,y) = F(x_0,y) + \int_{x_0}^{x} f(s,y)\,ds\,. \tag{4}$$

(b) Differentiate (4) with respect to y, then use (2) and the Fundamental Theorem of Calculus once more to obtain

$$\frac{\partial F}{\partial y} = g(x,y) - g(x_0,y) + \frac{dF(x_0,y)}{dy}\,. \tag{5}$$

(c) By requiring that

$$\frac{\partial F(x,y)}{\partial y} = g(x,y)\,, \tag{6}$$

deduce that

$$\frac{dF(x_0,y)}{dy} = g(x_0,y)\,.$$

Hence, use the Fundamental Theorem of Calculus again to obtain the following expression for $F(x_0,y)$:

$$F(x,y) = \int_{x_0}^{x} f(s,y)\,ds + \int_{y_0}^{y} g(x_0,t)\,dt + F(x_0,y_0)\,, \tag{7}$$

where y_0 is the initial point along the y-axis.

(d) By explicitly differentiating the expression for $F(x,y)$, verify that $\partial F/\partial x = f$ and $\partial F/\partial y = g$.

6.* Apply the result of Part 5(c) to each of the exact differentials in Part 4 to obtain the potential functions of those differentials.