First-Year Mathematics

Solutions to Problem Set 4 — January 28, 2005

1. In cylindrical coordinates,

$$x = r\cos\phi\,, \qquad y = r\sin\phi\,, \qquad z = z\,, \tag{1}$$

the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{2}$$

becomes

$$\frac{r^2\cos^2\phi}{a^2} + \frac{r^2\sin^2\phi}{a^2} + \frac{z^2}{b^2} = \frac{r^2}{a^2} + \frac{z^2}{b^2} = 1\,. \tag{3}$$

The ranges of r and ϕ are

$$0 \le r \le a\,, \qquad 0 \le \phi < 2\pi\,. \tag{4}$$

The range of z is bounded by the surface in Eq. (3) which, when solved for z is

$$z = b\left(1 - \frac{r^2}{a^2}\right)^{1/2}\,. \tag{5}$$

Hence, the range of z within the volume is

$$-b\left(1 - \frac{r^2}{a^2}\right)^{1/2} \le z \le b\left(1 - \frac{r^2}{a^2}\right)^{1/2}\,. \tag{6}$$

The volume bounded by V is thereby given by

$$\begin{aligned}
V &= \int_0^a r\,dr \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_{-b\sqrt{1-r^2/a^2}}^{b\sqrt{1-r^2/a^2}} dz}_{2b\sqrt{1-r^2/a^2}} \\
&= 4\pi b\int_0^a r\sqrt{1-r^2/a^2}\,dr \\
&= 4\pi b \underbrace{\left[-\frac{a^2}{3}\left(1-\frac{r^2}{a^2}\right)^{3/2}\Bigg|_0^a\right]}_{\frac{1}{3}a^2} \\
&= \frac{4}{3}\pi a^2 b\,.
\end{aligned} \tag{7}$$

Thus, in cylindrical coordinates,

$$\iiint_V z^2\,dx\,dy\,dz = \int_0^a r\,dr \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_{-b\sqrt{1-r^2/a^2}}^{b\sqrt{1-r^2/a^2}} z^2\,dz$$

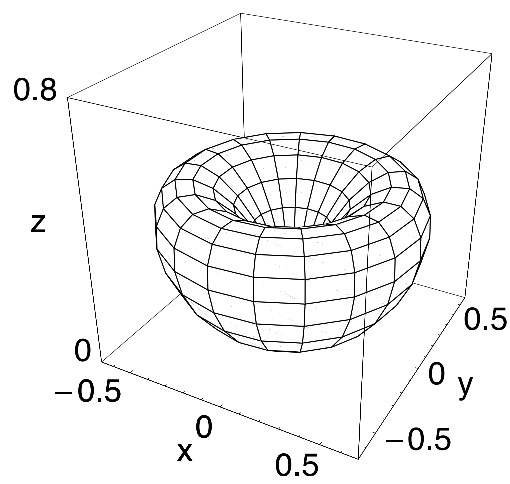
$$= 2\pi \int_0^a r\,dr \left(\frac{z^3}{3} \Big|_{-b\sqrt{1-r^2/a^2}}^{b\sqrt{1-r^2/a^2}} \right)$$

$$= \frac{4}{3}\pi b^3 \int_0^a r\left(1 - r^2/a^2\right)^{3/2} dr$$

$$= \frac{4}{3}\pi b^3 \underbrace{\left[-\frac{a^2}{5}\left(1 - \frac{r^2}{a^2}\right)^{5/2} \Bigg|_0^a \right]}_{\frac{1}{5}a^2}$$

$$= \frac{4}{15}\pi a^2 b^3\,. \tag{8}$$

2. The surface $(x^2+y^2+z^2)^2 = 2z(x^2+y^2)$ is shown below:



(a) In spherical polar coordinates

$$x = r\cos\phi\sin\theta\,, \qquad y = r\sin\phi\sin\theta\,, \qquad z = r\cos\phi\,, \tag{9}$$

the surface

$$(x^2+y^2+z^2)^2 = 2z(x^2+y^2) \tag{10}$$

becomes

$$r^4 = 2r\cos\theta(r^2\cos^2\phi\sin^2\theta + r^2\sin^2\phi\sin^2\theta) = 2r^3\cos\theta\sin^2\theta\,, \tag{11}$$

or,

$$r = 2\cos\theta\,\sin^2\theta\,. \tag{12}$$

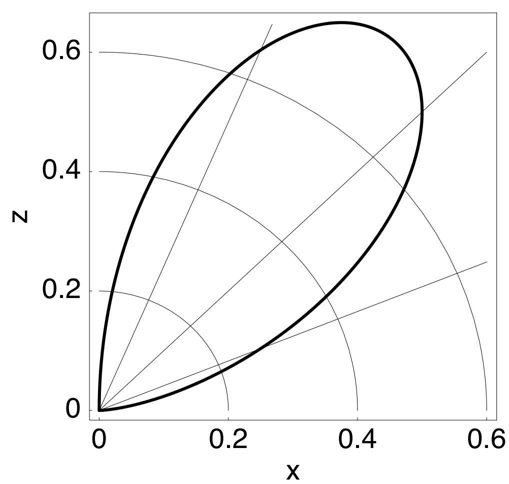
(b) The equation of the surface places no restrictions on ϕ, so this is a surface of revolution, which implies that

$$0 \le \phi < 2\pi\,. \tag{13}$$

Since r is a non-negative quantity, the right-hand side of Eq. (12) must also be non-negative. The factor of $\sin^2\phi$ provides no restriction on θ, but the factor of $\cos\theta$. To ensure that r is positive, we must restrict the range of θ to

$$0 \le \theta \le \tfrac{1}{2}\pi\,. \tag{14}$$

The cross-section of the surface in the x-z plane is shown below:



The lower bound of r is 0 and the upper bound of r is determined by the surface itself:

$$0 \le r \le 2\cos\theta\,\sin^2\theta\,. \tag{15}$$

(c) The volume enclosed by the surface is thus expressed as

$$V = \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_0^{\frac{1}{2}\pi} \sin\theta\,d\theta \int_0^{2\cos\theta\,\sin^2\theta} r^2\,dr$$

$$= 2\pi \int_0^{\frac{1}{2}\pi} \sin\theta \, d\theta \underbrace{\left(\frac{r^3}{3} \bigg|_0^{2\cos\theta\,\sin^2\theta} \right)}_{\frac{8}{3}\cos^3\theta\,\sin^6\theta}$$

$$= \frac{16\pi}{3} \int_0^{\frac{1}{2}\pi} \cos^3\theta \, \sin^7\theta \, d\theta \,. \tag{16}$$

Now, using the fact that $\cos^2\theta = 1 - \sin^2\theta$, the integrand in the last equation can be written as

$$\begin{aligned} V &= \frac{16\pi}{3} \int_0^{\frac{1}{2}\pi} \cos^2\theta \, \cos\theta \, \sin^7\theta \, d\theta \\ &= \frac{16\pi}{3} \int_0^{\frac{1}{2}\pi} (1 - \sin^2\theta) \, \cos\theta \, \sin^7\theta \, d\theta \\ &= \frac{16\pi}{3} \int_0^{\frac{1}{2}\pi} \cos\theta \, \sin^7\theta \, d\theta - \frac{2\pi}{3} \int_0^{\frac{1}{2}\pi} \cos\theta \, \sin^9\theta \, d\theta \\ &= \frac{16\pi}{3} \left(\frac{1}{8} \sin^8\theta \bigg|_0^{\frac{1}{2}\pi} \right) - \frac{16\pi}{3} \left(\frac{1}{10} \sin^1 0\theta \bigg|_0^{\frac{1}{2}\pi} \right) \\ &= \frac{16\pi}{3} \underbrace{\left(\frac{1}{8} - \frac{1}{10} \right)}_{\frac{1}{40}} = \frac{16\pi}{120} = \frac{2\pi}{15} \,. \end{aligned} \tag{17}$$

3. (a) The volume element of the spherical shell at (r, θ, ϕ) of extent dr, $d\theta$, and $d\phi$ is the incremental volume element in spherical polar coordinates:

$$r^2 \sin\theta \, dr \, d\theta \, d\phi \,. \tag{18}$$

Given that the mass density of the spherical shell has the uniform value ρ, the mass contained within this volume element is

$$\rho r^2 \sin\theta \, dr \, d\theta \, d\phi \,. \tag{19}$$

(b) The gravitational potential energy U between two point masses m and M separated by a distance r is

$$U = -\frac{GmM}{r} \tag{20}$$

The volume elements calculated in (a) which correspond to the same polar angle θ all lie at the same distance s from the point mass. Hence, the total potential energy dU between the corresponding masses and the point mass is obtained by integrating over the azimuthal angle ϕ:

$$dU = -\int_0^{2\pi} d\phi \left(\frac{Gm}{s} \rho\, r^2 \sin\theta\, dr\, d\theta \right) = -\frac{Gm}{s} (2\pi\rho\, r^2\, dr \sin\theta\, d\theta)\,. \tag{21}$$

By identifying the mass in the ring at polar angle θ as

$$dM = 2\pi\rho\, r^2\, dr \sin\theta\, d\theta\,, \tag{22}$$

we can write the gravitational potential energy between the mass in this ring and the point mass as

$$dU = -\frac{Gm\, dM}{s}\,. \tag{23}$$

(c) The volume of the spherical shell is $4\pi r^2\, dr$. The mass M contained within the shell is, therefore,

$$M = 4\pi r^2 \rho\, dr \tag{24}$$

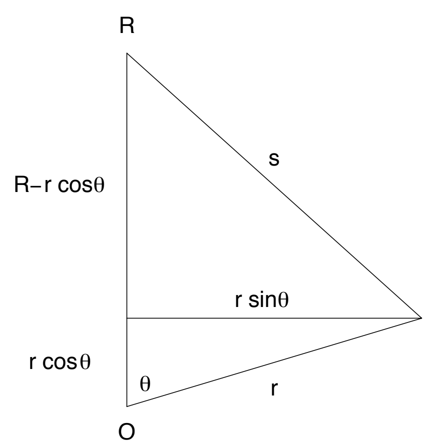
so the quantity dM calculated in Part (b) can be written in terms of M as

$$dM = \tfrac{1}{2} M \sin\theta\, d\theta\,. \tag{25}$$

Thus, the gravitational potential energy dU thus becomes

$$dU = -\frac{GmM \sin\theta\, d\theta}{2s}\,. \tag{26}$$

(d) Referring to the diagram below



we see that the distance s from the point mass at R to the shell is

$$\begin{aligned} s^2 &= (R - r\cos\theta)^2 + (r\sin\theta)^2 \\ &= R^2 - 2Rr\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta \\ &= R^2 - 2Rr\cos\theta + r^2\,. \end{aligned}$$

Thus, in the integral

$$U = -\tfrac{1}{2}GmM \int_0^{\pi} \frac{\sin\theta\, d\theta}{s(\theta)}\,, \tag{27}$$

changing variables from θ to s requires that we (i) determine the transformed integration element, (ii) change variables in the integrand, and (iii) change the limits of integration. Taking the differential of s^2, we obtain

(28)

which yields

$$d\theta = \frac{s\, ds}{rR\sin\theta}\,. \tag{29}$$

At $\theta = 0$, $s^2 = (R-r)^2$, so $s = R - r$ and at $\theta = \pi$, $s^2 = (R+r)^2$, so $s = R + r$. Thus, the integral determining U is transformed to

$$U = -\frac{GmM}{2rR}\int_{R-r}^{R+r} ds = -\frac{GmM}{R}\,. \tag{30}$$