First-Year Mathematics

Problem Set 4 January 28, 2005

1. Evaluate the integral
$$\iiint_V z^2\,dx\,dy\,dz\,,$$
over the volume V enclosed by the surface
$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2}=1\,,$$
where a and b are positive constants. Proceed by transforming the integral to cylindrical coordinates. Verify that you have the correct integration limits by calculating the volume V as
$$\iiint_V dx\,dy\,dz=\frac{4}{3}\pi a^2 b\,.$$
Hence, obtain
$$\iiint_V z^2\,dx\,dy\,dz=\frac{4}{15}\pi a^2 b^3\,.$$

2. Use spherical polar coordinates to determine the volume enclosed by the surface
$$(x^2+y^2+z^2)^2=2z(x^2+y^2)$$
by following the steps below:

   (a) Show that, in spherical polar coordinates, the equation of the surface is
   $$r=2\cos\theta\sin^2\theta\,.$$

   (b) To determine the ranges of integration, observe first that the equation places no restriction on the azimuthal angle ϕ, and that r must be a positive quantity. Hence, deduce that
   $$0\le\phi<2\pi\,,\qquad 0\le\theta\le\tfrac{1}{2}\pi\,.$$
   Finally, deduce the range of r from the equation of the surface bounding the volume.

   (c) Evaluate the radial and azimuthal integrals to obtain the volume V as
   $$V=\frac{16\pi}{3}\int_0^{\frac{1}{2}\pi}\cos^3\theta\,\sin^7\theta\,d\theta\,.$$
   Use the fact that $\cos^2\theta=1-\sin^2\theta$ to express this integral as the sum of two simpler integrals to obtain
   $$V=\frac{2\pi}{15}\,.$$

3. Consider a spherical shell of radius r and thickness dr that is centered at the origin and has a uniform mass density ρ. Suppose a point particle of mass m is placed at a distance R ($R > r$) from the origin. By following the steps below, determine the gravitational potential energy between the shell and the point mass.

   (a) Use a coordinate system whose z-axis coincides with the direction from the origin to the particle. For a *fixed* polar angle θ, consider a ring of width $d\theta$ and show that the volume corresponding to an infinitesimal variation $d\phi$ of the azimuthal angle is
   $$r^2 \sin\theta \, dr \, d\theta \, d\phi$$
   and that the mass contained within this elemental volume is
   $$\rho \, r^2 \sin\theta \, dr \, d\theta \, d\phi \, .$$

   (b) As ϕ is varied from 0 to 2π, the corresponding mass elements calculated in (a) all lie at the same distance s from the point mass. Hence, deduce that the gravitational potential energy dU between the ring at θ and the point mass is
   $$dU = -\frac{Gm \, dM}{s} \, ,$$
   where dM is the mass contained in the ring:
   $$dM = 2\pi \rho \, r^2 \, dr \sin\theta \, d\theta \, .$$

   (c) Show that the total mass M in the spherical shell is
   $$M = 4\pi r^2 \rho \, dr \, ,$$
   and, therefore, that the gravitational potential calculated in (b) is
   $$dU = -\frac{GmM \sin\theta \, d\theta}{2s} \, .$$

   (d) The distance s is a function of θ, $s = s(\theta)$, so the total gravitational potential energy U between the spherical shell and the point mass is
   $$U = -\tfrac{1}{2} GmM \int_0^\pi \frac{\sin\theta \, d\theta}{s(\theta)} \, .$$
   For a polar angle θ, show that the distance s from the shell to the point mass is
   $$s^2 = R^2 - 2rR\cos\theta + r^2 \, .$$
   Use this to change variables in the above integral from θ to s and show that U becomes
   $$U = -\frac{GmM}{2rR} \int_{R-r}^{R+r} ds = -\frac{GmM}{R} \, .$$
   This result demonstrates that the gravitational potential energy is equal to that of two point masses separated by a distance R. Thus, the effect of the spherical shell of mass has been replaced by that of a point particle situated at the origin with the same total mass!