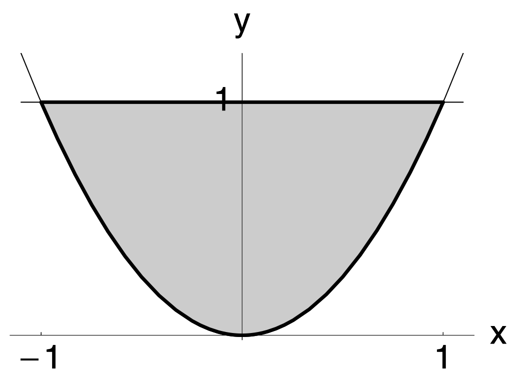
# First-Year Mathematics

Solutions to Problem Set 3 January 21, 2005

1. The region in the $x$-$y$ plane bounded by the line $y = 1$ and the parabola $y = x^2$ is shown shaded in the figure below:




(a) The area of this region is calculated in rectangular coordinates. By taking the range of $x$ as

$$-1 \le x \le 1\,, \tag{1}$$

the corresponding range of $y$ is bounded from below by the parabola $y = x^2$ and from above by the line $y = 1$:

$$x^2 \le y \le 1\,. \tag{2}$$

The area is thereby represented by the following double integral:

$$\begin{aligned} A = \int_{-1}^{1} dx \int_{x^2}^{1} dy &= \int_{-1}^{1} dx \left( y\Big|_{x^2}^{1} \right) \\ &= \int_{-1}^{1} dx - \int_{-1}^{1} x^2\,dx \\ &= \left( x\Big|_{-1}^{1} \right) - \left( \frac{x^3}{3}\Big|_{-1}^{1} \right) \\ &= 2 - \tfrac{2}{3} = \tfrac{4}{3}\,. \end{aligned} \tag{3}$$

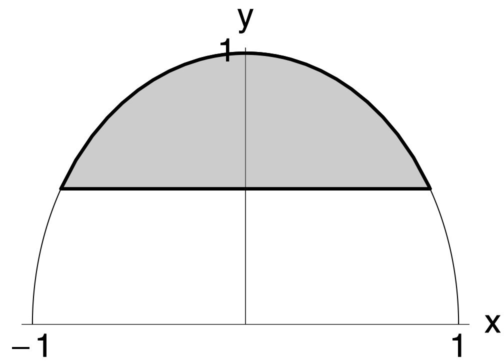
(b) The integral of $f(x,y) = x^2$ over this region is given by

$$\begin{aligned} \int_{-1}^{1} x^2\,dx \int_{x^2}^{1} dy &= \int_{-1}^{1} x^2\,dx \left( y\Big|_{x^2}^{1} \right) \\ &= \int_{-1}^{1} x^2(1 - x^2)\,dx \end{aligned}$$

$$= \left( \frac{x^3}{3} \Big|_{-1}^{1} \right) - \left( \frac{x^5}{5} \Big|_{-1}^{1} \right)$$

$$= \frac{2}{3} - \frac{2}{5} = \frac{4}{15} \,. \tag{4}$$

2. The integration region is bounded by the line $y = \frac{1}{2}$ and the unit semicircle $x^2 + y^2 = 1$, as shown in the figure below:




(a) The area of this region is calculated in rectangular coordinates. The calculation is simpler if the integral over $x$ is carried out before the integral over $y$. The range of $y$ is then given by

$$\tfrac{1}{2} \le y \le 1 \,, \tag{5}$$

and the corresponding range of $x$ is bounded from the left and right by the boundary of the unit circle:

$$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \,. \tag{6}$$

The area is now represented by the double integral

$$A = \int_{\frac{1}{2}}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx$$

$$= \int_{\frac{1}{2}}^{1} dy \left( x \Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \right)$$

$$= 2 \int_{\frac{1}{2}}^{1} \sqrt{1-y^2} \, dy \,. \tag{7}$$

This integral is evaluated by trigonometric substitution. Accordingly, we change variables to $y = \sin t$. Then,

$$\sqrt{1-y^2} \, dy = \sqrt{1-\sin^2 t} \cos t \, dt = \cos^2 t \, dt \,, \tag{8}$$

and

$$y = \tfrac{1}{2} \longrightarrow t = \tfrac{1}{6}\pi\,, \qquad y = 1 \longrightarrow t = \tfrac{1}{2}\pi\,. \tag{9}$$

Thus, the integral becomes

$$\begin{aligned} A &= 2\int_{\frac{1}{2}}^{1} \sqrt{1-y^2}\,dy = 2\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \cos^2 t\,dt \\ &= \left(\tfrac{1}{2}\pi - \tfrac{1}{6}\pi\right) + \left[\sin\left(\tfrac{1}{2}\pi\right)\cos\left(\tfrac{1}{2}\pi\right) - \sin\left(\tfrac{1}{6}\pi\right)\cos\left(\tfrac{1}{6}\pi\right)\right] \\ &= \tfrac{1}{3}\pi - \tfrac{1}{4}\sqrt{3}\,, \end{aligned} \tag{10}$$

where we have used the result in Part 6 of Classwork 1.

(b) The integral of $f(x, y) = x + y$ over this region is written as the sum of two integrals:

$$\begin{aligned} \int_{\frac{1}{2}}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx(x+y) &= \int_{\frac{1}{2}}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x\,dx + \int_{\frac{1}{2}}^{1} y\,dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx \\ &= \int_{\frac{1}{2}}^{1} dy \underbrace{\left(\frac{x^2}{2}\bigg|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\right)}_{0} + 2\int_{\frac{1}{2}}^{1} y\sqrt{1-y^2}\,dy \\ &= 2\int_{\frac{1}{2}}^{1} y\sqrt{1-y^2}\,dy\,. \end{aligned} \tag{11}$$

Note that the first integral vanishes because the region is symmetric about the $y$-axis and $x$ is an odd function. To evaluate the remaining integral we need to determine the anti-derivative of the integrand. This is accomplished by observing that, since

$$\frac{d}{dy}(1-y^2)^{3/2} = \tfrac{3}{2}(1-y^2)^{1/2}(-2y) = -3y(1-y^2)^{1/2}\,, \tag{12}$$

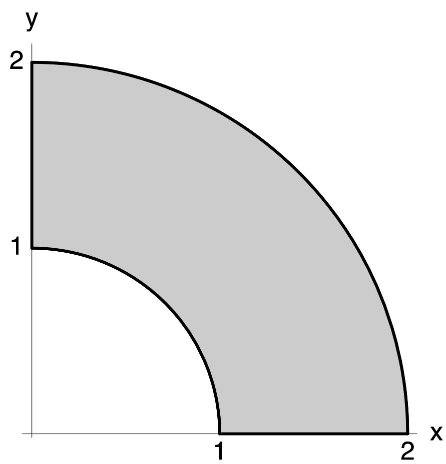
the anti-derivative of $2\sqrt{1-y^2}$ is $-\frac{2}{3}(1-y^2)^{3/2}$, i.e.

$$\frac{d}{dy}\left[-\tfrac{2}{3}(1-y^2)^{3/2}\right] = 2\sqrt{1-y^2}\,, \tag{13}$$

Hence, the integral of $f$ over the region is

$$2\int_{\frac{1}{2}}^{1} y\sqrt{1-y^2}\,dy = -\tfrac{2}{3}(1-y^2)^{3/2}\bigg|_{\frac{1}{2}}^{1} = \tfrac{2}{3}\left(\tfrac{3}{4}\right)^{3/2}\,. \tag{14}$$

3. To evaluate the function $f(x, y) = xy$ over the shaded region shown below,




we use polar coordinates. The range of $r$ within this region is bounded by the circles with radii 1 and 2, so

$$1 \le r \le 2\,. \tag{15}$$

The range of $\phi$ is between the $x$-axis ($\phi = 0$) and the $y$-axis ($y = \frac{1}{2}\pi$):

$$0 \le \phi \le \tfrac{1}{2}\pi\,. \tag{16}$$

In polar coordinates $x = r\cos\phi$ and $y = r\sin\phi$, so

$$f(x, y) = xy = r\cos\phi\, r\sin\phi = r^2\cos\phi\sin\phi\,. \tag{17}$$

Thus, the integral of $f$ over the region is

$$\int_1^2 r\,dr \int_0^{\frac{1}{2}\pi} d\phi\, r^2\cos\phi\sin\phi = \int_1^2 r^3\,dr \int_0^{\frac{1}{2}\pi} \cos\phi\sin\phi\,d\phi\,. \tag{18}$$

These two integrals can be evaluated independently:

$$\int_0^{\frac{1}{2}\pi} \cos\phi\sin\phi\,d\phi = \frac{1}{2}\sin^2\phi\bigg|_0^{\frac{1}{2}\pi} = \frac{1}{2}\,, \tag{19}$$

and

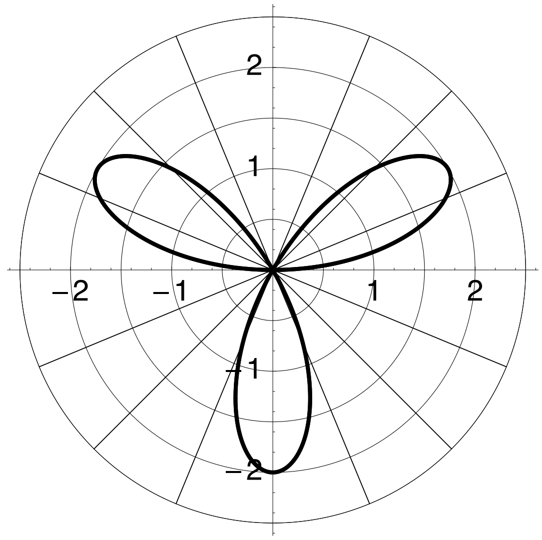
$$\int_1^2 r^3\,dr = \frac{r^4}{4}\bigg|_1^2 = 4 - \frac{1}{4} = \frac{15}{4}\,. \tag{20}$$

Therefore,

$$\int_1^2 r\,dr \int_0^{\frac{1}{2}\pi} d\phi r^2\cos\phi\sin\phi = \frac{15}{4} \times \frac{1}{2} = \frac{15}{8}\,. \tag{21}$$

Notice that, if the range $\phi$ is extended to $\pi$, the integral of $f$ over the resulting region vanishes, again because of symmetry, as discussed in Part 2(b).

4. To determine the ranges of $r$ and $\phi$ within the petal that lies in the first quadrant in the graph $r = 2\sin(3\phi)$, shown below,




we first observe that $r(0) = 0$ and that $r$ returns to the origin first for $\phi = \frac{1}{3}\pi$. Thus, the range of $\phi$ for the area within the first petal is

$$0 \le \phi \le \tfrac{1}{3}\pi\,. \tag{22}$$

The corresponding range of $r$ is obtained by noting that the lower bound is $r = 0$ and the upper bound is the curve of the graph, i.e. $r = 2\sin(3\phi)$. Thus, the range of $r$ is

$$0 \le r \le 2\sin(3\phi)\,, \tag{23}$$

and the area $A$ of the petal can be represented as a double integral in polar coordinates:

$$A = \int_0^{\frac{1}{3}\pi} d\phi \int_0^{2\sin(3\phi)} r\,dr\,. \tag{24}$$

This integral is evaluated as follows:

$$\begin{aligned} A &= \int_0^{\frac{1}{3}\pi} d\phi \int_0^{2\sin(3\phi)} r\,dr = \int_0^{\frac{1}{3}\pi} \sin(3\phi)\,d\phi \left( \frac{r^2}{2}\bigg|_0^{2\sin(3\phi)} \right) \\ &= 4\int_0^{\frac{1}{3}\pi} \sin^2(3\phi)\,d\phi \end{aligned} \tag{25}$$

If we change the integration variable to $t = 3\phi$, then

$$\sin^2(3\phi)\,d\phi = \tfrac{1}{3}\sin^2 t\,dt\,, \tag{26}$$

and

$$\phi = 0 \longrightarrow t = 0\,, \qquad \phi = \tfrac{1}{3}\pi \longrightarrow t = \pi\,, \tag{27}$$

so that the area integral is transformed to

$$A = 4\int_0^{\frac{1}{3}\pi} \sin^2(3\phi)\,d\phi = \frac{4}{3}\int_0^{\pi} \sin^2 t\,dt = \frac{2\pi}{3}\,. \tag{28}$$