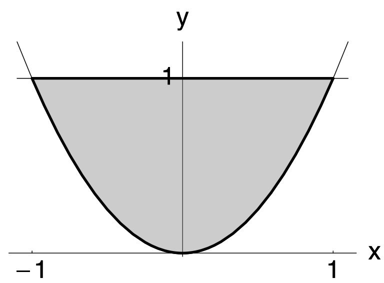
First-Year Mathematics

Problem Set 3 January 21, 2005

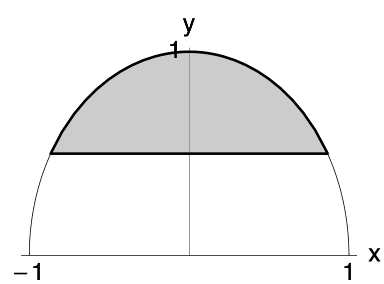
1. Consider the region in the x-y plane bounded by the line $y = 1$ and the parabola $y = x^2$, shown shaded in the figure below:



   (a) Compute the area of this region by using a double integral.

   (b) Integrate the function $f(x, y) = x^2$ over this region.

2. Consider the region in the x-y plane bounded by the line $y = \frac{1}{2}$ and the unit semicircle $x^2 + y^2 = 1$, shown shaded in the figure below:

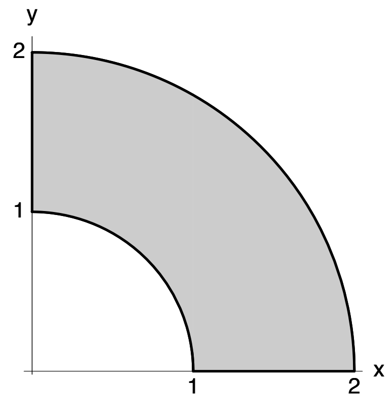


   (a) Compute the area of this region by using a double integral. The calculation is simpler if the integral over x is performed before the integral over y, i.e. by representing the area as

$$A = \int_{\frac{1}{2}}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,.$$

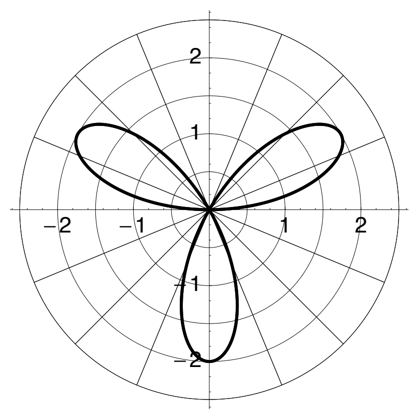
   (b) Integrate the function $f(x, y) = x + y$ over this region.

3. Consider the sub-region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$ shown below:



Compute the value of $f(x, y) = xy$ over this region using polar coordinates.

4. Determine the area enclosed by one "petal" of the graph of $r = 2\sin(3\phi)$, which is shown below:



This is the same type of exercise as Part 3 of Classwork 3. The areas of all three petals are the same (because of the symmetry of the graph), so we can focus on any one of these. Consider the petal in the first quadrant. Show that the range of ϕ is

$$0 \leq \phi \leq \tfrac{1}{3}\pi\,,$$

and that the corresponding range of r is

$$0 \leq r \leq 2\sin(3\phi)\,.$$

Hence, the area A of the petal in the first quadrant is

$$A = 2\int_0^{\frac{1}{3}\pi} d\phi \int_0^{2\sin(3\phi)} r\,dr\,.$$

Evaluate this integral to obtain

$$A = \frac{2\pi}{3}\,.$$