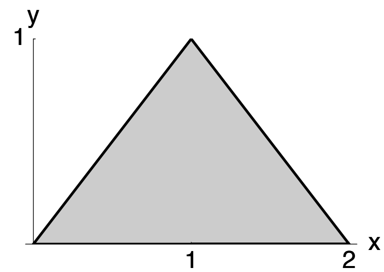
First-Year Mathematics

Solutions to Problem Set 2 January 14, 2005

1. The region over which the integral is taken is shown below:



(a) If we allow x to vary between 0 and 2, then the range of y is restricted by the curves that bound the region. In the interval $0 \leq x \leq 1$, y is bounded from below by the x-axis and from above by the curve $y = x$, so the range of y is

$$0 \leq y \leq x\,. \tag{1}$$

In the interval $1 \leq x \leq 2$, y is bounded from below by the x-axis and from above by the curve $y = 2 - x$, so the range of y is

$$0 \leq y \leq 2 - x\,. \tag{2}$$

The integral to be evaluated is thereby

$$\begin{aligned}
\iint_A xy\,dx\,dy &= \int_0^1 x\,dx \int_0^x y\,dy + \int_1^2 x\,dx \int_0^{2-x} y\,dy \\
&= \int_0^1 x\,dx \left(\tfrac{1}{2}y^2\Big|_0^x\right) + \int_1^2 x\,dx \left(\tfrac{1}{2}y^2\Big|_0^{2-x}\right) \\
&= \tfrac{1}{2}\int_0^1 x^3\,dx + \tfrac{1}{2}\int_1^2 x \underbrace{(2-x)^2}_{4-4x+x^2} dx \\
&= \tfrac{1}{2}\int_0^2 x^3\,dx + 2\int_1^2 (x - x^2)\,dx \\
&= \tfrac{1}{8}x^4\Big|_0^2 + x^2\Big|_1^2 - \tfrac{2}{3}x^3\Big|_1^2 \\
&= 2 + (4-1) - \tfrac{2}{3}(8-1) = \tfrac{1}{3}\,.
\end{aligned} \tag{3}$$

(b) If we allow y to vary between 0 and 1, the range of x is bounded on the left by the straight line $y = x$ and on the right by the straight line $y = 2 - x$. Thus, the range of x is

$$y \leq x \leq 2 - y\,. \tag{4}$$

The integral to be evaluated is

$$\begin{aligned}
\iint_A xy\,dx\,dy &= \int_0^1 y\,dy \int_y^{2-y} x\,dx \\
&= \int_0^1 y\,dy \left(\tfrac{1}{2}x^2 \Big|_y^{2-y} \right) \\
&= \tfrac{1}{2} \int_0^1 y \underbrace{\left[(2-y)^2 - y^2\right]}_{4-4y} dy \\
&= \left(y^2 - \tfrac{2}{3}\right) \Big|_0^1 \\
&= \tfrac{1}{3}\,.
\end{aligned} \tag{5}$$

2. The region A is the interior of the curve $x^2 + y^2 = R^2$. Thus, for a given value of x,

$$-\sqrt{R^2 - x^2} \leq y \leq \sqrt{R^2 - x^2}\,, \tag{6}$$

and, for a given value of y,

$$-\sqrt{R^2 - y^2} \leq x \leq \sqrt{R^2 - y^2}\,. \tag{7}$$

Thus, the interior of this circular region can be represented by either of the following integrals:

$$\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \qquad \text{or} \qquad \int_{-R}^{R} dy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dx\,. \tag{8}$$

Evaluating the first of these, we have

$$\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy = \int_{-R}^{R} dx \left\{ y \Big|_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \right\} = 2 \int_{-R}^{R} \sqrt{R^2 - x^2}\,dx\,. \tag{9}$$

This is a standard example of an integral whose evaluation proceeds by trigonometric substitution. With $x = R\sin\phi$, we have

$$\sqrt{R^2 - x^2} = \sqrt{R^2 - R^2\sin^2\phi} = R\sqrt{1 - \sin^2\phi} = R\cos\phi\,, \tag{10}$$

and

$$dx = R\cos\phi\,d\phi\,. \tag{11}$$

Then, with $x = -R$ corresponding to $\phi = -\frac{1}{2}\pi$ and $x = R$ to $\phi = \frac{1}{2}\pi$, our integral is transformed to

$$2R^2 \underbrace{\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^2\phi\, d\phi}_{\frac{1}{2}\pi} = \pi R^2\,. \tag{12}$$

3. To evaluate the integral in Part 2 in circular polar coordinates, we first determine the ranges of the integration variables. For the interior of a circle of radius R, we have that

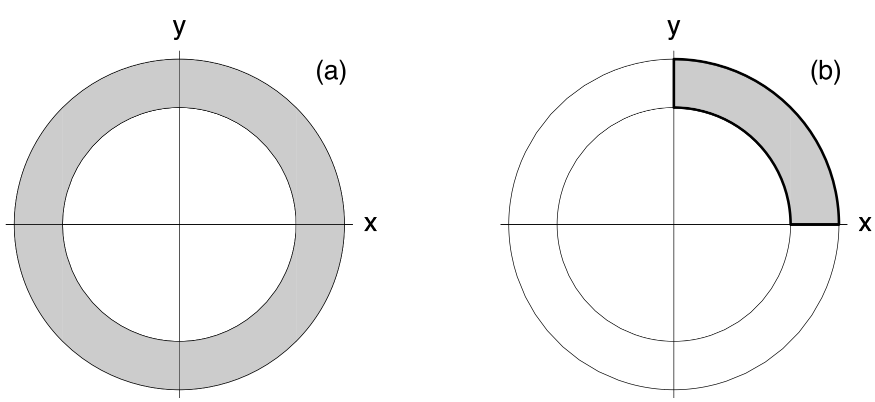
$$0 \le r \le R\,, \qquad 0 \le \phi < 2\pi\,. \tag{13}$$

The integral to be evaluated is, therefore,

$$\begin{aligned} A &= \int_0^R r\,dr \int_0^{2\pi} d\phi \\ &= \left(\tfrac{1}{2}r^2\Big|_0^R\right)\left(\phi\Big|_0^{2\pi}\right) \\ &= \tfrac{1}{2}R^2 \times 2\pi \\ &= \pi R^2\,, \end{aligned} \tag{14}$$

which is the area of the circle.

4. To determine the area between two circles of radii a and b, where $b > a$, in circular polar coordinates, we refer to figure (a) below:



The region to be integrated, which is shaded, corresponds to the following ranges of r and ϕ:

$$a \le r \le b\,, \qquad 0 \le \phi < 2\pi\,. \tag{15}$$

The corresponding integral is

$$\int_a^b r\,dr \int_0^{2\pi} d\phi = 2\pi \left(\tfrac{1}{2} r^2 \Big|_a^b \right) = \pi(b^2 - a^2)\,, \tag{16}$$

which can be interpreted as the area of the larger circle, πb^2, minus the area of the smaller circle, πa^2.

The area of the shaded region in figure (b) above is similarly calculated. The ranges of r and ϕ are

$$a \le r \le b\,, \qquad 0 \le \phi \le \tfrac{1}{2}\pi\,, \tag{17}$$

and the integral is

$$\int_a^b r\,dr \int_0^{\frac{1}{2}\pi} d\phi = \tfrac{1}{2}\pi \left(\tfrac{1}{2} r^2 \Big|_a^b \right) = \tfrac{1}{4}\pi(b^2 - a^2)\,. \tag{18}$$

5. (a) The Cartesian representation of the vector $\boldsymbol{r}$ that points from the origin to the point (x, y) are $\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$. In polar coordinates, the CArtesian components of this vector are

$$x = r\cos\phi\,, \qquad y = r\sin\phi\,, \tag{19}$$

and this vector is expressed as

$$\boldsymbol{r} = r\cos\phi\,\boldsymbol{i} + r\sin\phi\,\boldsymbol{j}\,. \tag{20}$$

The lines of constant $r = |\boldsymbol{r}| = (x^2 + y^2)^{1/2}$ are circles of radius r centered at the origin.

(b) The tangent to the circles of constant r are determined by taking the derivative of $\boldsymbol{r}$ with respect to ϕ. As discussed in Sec. 1.1, this requires differentiating each of the components of $\boldsymbol{r}$:

$$\boldsymbol{t} = \frac{\partial \boldsymbol{r}}{\partial \phi} = -r\sin\phi\,\boldsymbol{i} + r\cos\phi\,\boldsymbol{j}\,. \tag{21}$$

Notice that, in Cartesian coordinates (x, y), this vector can be written as

$$\boldsymbol{t} = -y\,\boldsymbol{i} + x\,\boldsymbol{j}\,. \tag{22}$$

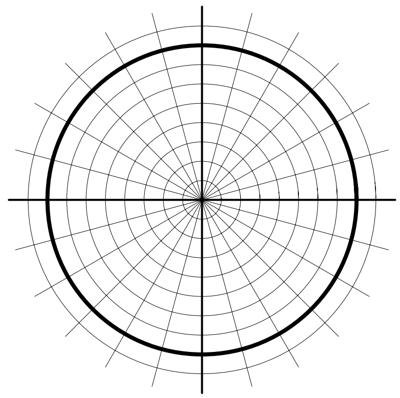
(c) The "dot" product $\boldsymbol{r} \cdot \boldsymbol{t}$ is

$$\begin{aligned} \boldsymbol{r} \cdot \boldsymbol{t} &= (r\cos\phi\,\boldsymbol{i} + r\sin\phi\,\boldsymbol{j}) \cdot (-r\sin\phi\,\boldsymbol{i} + r\cos\phi\,\boldsymbol{j}) \\ &= -r^2 \sin\phi\cos\phi + r^2\sin\phi\cos\phi \\ &= 0\,, \end{aligned} \tag{23}$$

or, in terms of Cartesian coordinates,

$$(x\,\boldsymbol{i} + y\,\boldsymbol{j}) \cdot (-y\,\boldsymbol{i} + x\,\boldsymbol{j}) = -xy + xy = 0\,, \tag{24}$$

which demonstrates the orthogonality of circular polar coordinates. This can be seen directly from the coordinate curves in the circular polar cooordinate system, as shown below (Fig. 2.8(b) from the course notes):



6. Beginning with

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx\,, \tag{25}$$

we write the square of this integral as

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-(x^2+y^2)}\,. \tag{26}$$

We must use *different* integration variables for the two integrations in I^2 because the integration variables range between $-\infty$ and ∞ *independently.* Thus, since these two integrations are combined into a single double integral, we must, for book-keeping purposes, use two different variables. Now, viewed as a double integral, the region of integration is the entire x-y plane. This integral will be evaluated by first transforming to circular polar coordinates,

$$x = r\cos\phi\,, \qquad y = r\sin\phi\,, \tag{27}$$

with

$$0 \le r < \infty\,, \qquad 0 \le \phi < 2\pi\,, \tag{28}$$

in which case the integral becomes

$$I^2 = \int_0^r r\,e^{-r^2}\,dr \underbrace{\int_0^{2\pi} d\phi}_{2\pi} = 2\pi \int_0^r r\,e^{-r^2}\,dr\,. \tag{29}$$

The radial integral is carried out as follows

$$\int_0^r r\, e^{-r^2}\, dr = -\tfrac{1}{2} e^{-r^2}\Big|_0^\infty = -\tfrac{1}{2}\left(\lim_{r\to\infty} e^{-r^2} - 1\right) = \tfrac{1}{2}\,. \tag{30}$$

Thus, $I^2 = \pi$ or,

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}\,. \tag{31}$$