First-Year Mathematics

Problem Set 2 January 14, 2005

1. Evaluate the double integral
$$\iint_A xy\,dx\,dy\,,$$
where the region A is the triangle with vertices at $(0,0)$, $(1,1)$, and $(2,0)$, in the two ways outlined below:

   (a) Allow x to vary between 0 and 2 and consider the corresponding allowed values of y. The intervals $0 \le x \le 1$ and $1 \le x \le 2$ must be done separately. Hence, obtain the *two* integrals:
$$\iint_A xy\,dx\,dy = \int_0^1 x\,dx \int_0^x y\,dy + \int_1^2 x\,dx \int_0^{2-x} y\,dy\,.$$

   (b) Now allow y to vary between 0 and 1 and consider the corresponding allowed values of x. This results in a *single* integral:
$$\iint_A xy\,dx\,dy = \int_0^1 y\,dy \int_y^{2-y} x\,dx\,.$$

   Ans: $\frac{1}{3}$.

2. Evaluate the double integral
$$\iint_A dx\,dy\,,$$
where A is the interior of a circle of radius R in rectangular coordinates The final step in the calculation requires evaluating the integral
$$2\int_{-R}^{R} \sqrt{R^2 - s^2}\,ds\,,$$
where s is either x or y (depending on whether the y or x integration is carried out first). Use the trigonometric substitution $s = R\sin\phi$ to transform this integral to
$$2R^2 \int_{-\pi/2}^{\pi/2} \cos^2\phi\,d\phi$$
and obtain
$$\iint_A dx\,dy = \pi R^2\,.$$

3. Evaulate the integral in Part 2 by using circular polar coordinates.

4. Suppose there are two circles of radius a and b, with $b > a$, both of which are centered at the origin. Use integration with polar coordinates to determine (a) the area between these circles, and (b) the area bounded by these circles between $\phi = 0$ and $\phi = \frac{1}{2}\pi$.

   Ans: (a) $\pi(b^2 - a^2)$, (b) $\frac{1}{4}\pi(b^2 - a^2)$.

5. One of the most important properties of polar coordinates (r, ϕ) is *orthogonality*, i.e. the property that lines of constant r intersect lines of constant ϕ at right angles. For this reason, polar coordinates are referred to as *orthogonal* coordinates. To demonstrate this property, consider any point (r, ϕ). Then,

   (a) Determine the Cartesian components of the radial vector $\boldsymbol{r}$ that lies along the line from the origin to (r, ϕ) and points in the direction of increasing r.

   (b) Determine the vector $\boldsymbol{t}$ tangent to the circle at (r, ϕ). (Think of a particle moving along a circular path with ϕ as the "time". The vector $\boldsymbol{t}$ is the "velocity" of this particle.)

   (c) Calculate the "dot" product $\boldsymbol{r} \cdot \boldsymbol{t}$.

   Ans: (a) $\boldsymbol{r} = r\cos\phi\,\boldsymbol{i} + r\sin\phi\,\boldsymbol{j}$, (b) $\boldsymbol{t} = -r\sin\phi\,\boldsymbol{i} + r\cos\phi\,\boldsymbol{j}$, (c) $\boldsymbol{r} \cdot \boldsymbol{t} = 0$.

6. A novel use of polar coordinates is for the evaluation of the following integral:

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx\,.$$

   This integral is encountered in quantum mechanics, statistical mechanics, and probability theory. Proceed by squaring this integral and writing the product as

$$I^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \mathrm{e}^{-x^2-y^2}\,.$$

   Notice that we have introduced a second integration variable (why?). Transform this integral into polar coordinates and evaluate the resulting integrals over r and ϕ to deduce that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}\,.$$