First-Year Mathematics

Solutions to Problem Set 1 January 7, 2005

1. The definition of the derivative of a function $f(x)$ is

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right] . \tag{1}$$

(a) $f = x^3$. From the definition in Eq. (1), we have

$$\begin{aligned} \frac{d(x^3)}{dx} &= \lim_{\Delta \to 0} \left[\frac{(x+\Delta x)^3 - x^3}{\Delta x} \right] \\ &= \lim_{\Delta x \to 0} \left[\frac{(x^3 + 3x^2 \Delta x + \cdots) - x^3}{\Delta x} \right] \\ &= \lim_{\Delta x \to 0} \left[\frac{3x^2 \Delta x + \cdots}{\Delta x} \right] \\ &= 3x^2 , \end{aligned} \tag{2}$$

where "$\cdots$" signifies terms that are of higher order in Δx, which vanish in the limit $\Delta x \to 0$.

(b) $f = x^{1/2}$. From the definition in Eq. (1), we have

$$\frac{d(x^{1/2})}{dx} = \lim_{\Delta x \to 0} \left[\frac{(x+\Delta x)^{1/2} - x^{1/2}}{\Delta x} \right] . \tag{3}$$

Applying the binomial series to the term $(x+\Delta x)^{1/2}$ and retaining terms only to first order in Δx yields

$$\begin{aligned} (x+\Delta x)^{1/2} &= x^{1/2} \left(1 + \frac{\Delta x}{x} \right)^{1/2} \\ &= x^{1/2} \left(1 + \frac{\Delta x}{2x} + \cdots \right) \\ &= x^{1/2} + \tfrac{1}{2} x^{-1/2} \Delta x \cdots . \end{aligned} \tag{4}$$

By substituting this expression into Eq. (3), we obtain

$$\begin{aligned}\frac{d(x^{1/2})}{dx} &= \lim_{\Delta x\to 0}\left[\frac{(x^{1/2}+\frac{1}{2}x^{-1/2}\Delta x)+\cdots - x^{1/2}}{\Delta x}\right]\\ &= \lim_{\Delta x\to 0}\left[\frac{\frac{1}{2}x^{-1/2}\Delta x+\cdots}{\Delta x}\right]\\ &= \tfrac{1}{2}x^{-1/2}\,. \end{aligned} \tag{5}$$

(c) $f = x^{-1/2}$. From the definition in Eq. (1), we have

$$\frac{d(x^{-1/2})}{dx} = \lim_{\Delta x\to 0}\left[\frac{(x+\Delta x)^{-1/2} - x^{-1/2}}{\Delta x}\right]\,. \tag{6}$$

Applying the binomial series to the term $(x+\Delta x)^{-1/2}$ and retaining terms only to first order in Δx yields

$$\begin{aligned}(x+\Delta x)^{-1/2} &= x^{-1/2}\left(1+\frac{\Delta x}{x}\right)^{-1/2}\\ &= x^{-1/2}\left(1-\frac{\Delta x}{2x}+\cdots\right)\\ &= x^{-1/2} - \tfrac{1}{2}x^{-3/2}\Delta x\cdots\,. \end{aligned} \tag{7}$$

By substituting this expression into Eq. (6), we obtain

$$\begin{aligned}\frac{d(x^{-1/2})}{dx} &= \lim_{\Delta x\to 0}\left[\frac{(x^{-1/2}-\frac{1}{2}x^{-3/2}\Delta x)+\cdots - x^{-1/2}}{\Delta x}\right]\\ &= \lim_{\Delta x\to 0}\left[\frac{-\frac{1}{2}x^{-3/2}\Delta x+\cdots}{\Delta x}\right]\\ &= -\tfrac{1}{2}x^{-3/2}\,. \end{aligned} \tag{8}$$

2. The derivative of a composite function $f(g(x))$, where f and g are differentiable functions, is defined as

$$\frac{d\,f(g)}{dx} \equiv \lim_{\Delta x\to 0}\left[\frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x}\right]\,. \tag{9}$$

This expression can be written as

$$\frac{d\,f(g)}{dx} \equiv \lim_{\Delta x\to 0}\left\{\left[\frac{f(g(x+\Delta x)) - f(g(x))}{g(x+\Delta x)-g(x)}\right]\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]\right\}\,. \tag{10}$$

By defining $\Delta g = g(x+\Delta x) - g(x)$, so $g(x+\Delta) = g(x) + \Delta g$, we can write

$$\frac{d\,f(g)}{dx} \equiv \lim_{\Delta x \to 0} \left\{ \left[\frac{f(g(x)+\Delta g) - f(g(x))}{\Delta g} \right] \left[\frac{g(x+\Delta x) - g(x)}{\Delta x} \right] \right\} . \tag{11}$$

Since g is a continuous function (because it is differentiable), $\Delta g \to 0$ as $\Delta x \to 0$. Thus, since

$$\frac{df}{dg} \equiv \lim_{\Delta g \to 0} \left[\frac{f(g(x)+\Delta g) - f(g(x))}{\Delta g} \right] , \tag{12}$$

and

$$\frac{dg}{dx} \equiv \lim_{\Delta x \to 0} \left[\frac{g(x+\Delta x) - g(x)}{\Delta x} \right] , \tag{13}$$

we arrive at the chain rule:

$$\frac{d\,f(g)}{dx} = \frac{df}{dg}\frac{dg}{dx} . \tag{14}$$

3. (a) $f(x,y) = \sqrt{x^2+y^2}$. The two first partial derivatives of f are calculated as

$$\frac{\partial}{\partial x}(x^2+y^2)^{1/2} = \tfrac{1}{2}(x^2+y^2)^{-1/2} \times 2x = x(x^2+y^2)^{-1/2} , \tag{15}$$

$$\frac{\partial}{\partial y}(x^2+y^2)^{1/2} = \tfrac{1}{2}(x^2+y^2)^{-1/2} \times 2y = y(x^2+y^2)^{-1/2} . \tag{16}$$

(b) $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$. The three first partial derivatives of f are calculated as

$$\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = -\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2} \times 2x = -x(x^2+y^2+z^2)^{-3/2} , \tag{17}$$

$$\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} = -\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2} \times 2y = -y(x^2+y^2+z^2)^{-3/2} , \tag{18}$$

$$\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2} = -\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2} \times 2z = -z(x^2+y^2+z^2)^{-3/2} . \tag{19}$$

(c) $f(x,y) = \ln(xy)$. The two first partial derivatives of f are calculated as

$$\frac{\partial}{\partial x}\ln(xy) = \frac{1}{xy} \times y = \frac{1}{x} , \tag{20}$$

$$\frac{\partial}{\partial y}\ln(xy) = \frac{1}{xy} \times x = \frac{1}{y} . \tag{21}$$

(d) $f = e^{x/y}$. The two first partial derivatives of f are calculated as

$$\frac{\partial}{\partial x} = e^{x/y} \times \frac{1}{y} = \frac{e^{x/y}}{y}\,, \tag{22}$$

$$\frac{\partial}{\partial y} = e^{x/y} \times \left(-\frac{x}{y^2}\right) = -\frac{x e^{x/y}}{y^2}\,. \tag{23}$$

4. From the definition of the derivative in Eq. (1),

$$\frac{d(\sin x)}{dx} = \lim_{\Delta x \to 0} \left[\frac{\sin(x + \Delta x) - \sin x}{\Delta x}\right]. \tag{24}$$

By using the identity for the sine of the sum of two angles, we have

$$\sin(x + \Delta x) = \sin x \cos(\Delta x) + \cos x \sin(\Delta x) = \sin x + \Delta x \cos x + \cdots\,. \tag{25}$$

Substitution of this expression into Eq. (24) yields

$$\begin{aligned} \frac{d(\sin x)}{dx} &= \lim_{\Delta x \to 0} \left[\frac{(\sin x + \Delta x \cos x) - \sin x}{\Delta x}\right] \\ &= \cos x\,. \end{aligned} \tag{26}$$

Similarly,

$$\frac{d(\cos x)}{dx} = \lim_{\Delta x \to 0} \left[\frac{\cos(x + \Delta x) - \cos x}{\Delta x}\right]. \tag{27}$$

By using the identity for the cosine of the sum of two angles, we have

$$\cos(x + \Delta x) = \cos x \cos(\Delta x) - \sin x \sin(\Delta x) = \cos x - \Delta x \sin x + \cdots\,. \tag{28}$$

Substitution of this expression into Eq. (27) yields

$$\begin{aligned} \frac{d(\cos x)}{dx} &= \lim_{\Delta x \to 0} \left[\frac{(\cos x - \Delta x \sin x) - \cos x}{\Delta x}\right] \\ &= -\sin x\,. \end{aligned} \tag{29}$$

5. The Fundamental Theorem of Calculus states that

$$\int_a^b f(x)\,dx = F(b) - F(a)\,, \tag{30}$$

where $dF/dx = f$. Similarly, by reversing the limits of integration, we have

$$\int_b^a f(x)\,dx = F(a) - F(b)\,. \tag{31}$$

By comparing the right-hand sides of Eqs. (30) and (31), we conclude that

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx\,. \tag{32}$$

6. The Fundamental Theorem of Calculus states that

$$\int_a^x f(s)\,ds = F(x) - F(a)\,, \tag{33}$$

where $dF/dx = f$. By differentiating both sides with respect to x, we obtain,

$$\frac{d}{dx}\left[\int_a^x f(s)\,ds\right] = \frac{dF}{dx} = f(x)\,. \tag{34}$$

Similarly, by writing the Fundamental Theorem of Calculus as

$$\int_x^b f(s)\,ds = F(b) - F(x)\,, \tag{35}$$

and differentiating with respect to x, we obtain

$$\frac{d}{dx}\left[\int_x^b f(s)\,ds\right] = -\frac{dF}{dx} = -f(x)\,. \tag{36}$$

Finally, by writing the Fundamental Theorem of Calculus as

$$\int_{u(x)}^{v(x)} f(s)\,dx = F[v(x)] - F[u(x)]\,, \tag{37}$$

and differentiating with respect to x using the chain rule, we obtain

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(s)\,dx\right] = \frac{dF}{dv}\frac{dv}{dx} - \frac{dF}{du}\frac{du}{dx} = f[v(x)]\frac{dv}{dx} - f[u(x)]\frac{du}{dx}\,. \tag{38}$$

7. The Fundamental Theorem of Calculus states that

$$\int_a^x f(s)\,ds = F(x) - F(a)\,, \tag{39}$$

where

$$\frac{dF}{dx} = f\,. \tag{40}$$

From Part 6 of Classwork 1, we set $b = x$, so that we can write,

$$\begin{aligned}
\int_a^x \cos^2 s\, ds &= \tfrac{1}{2}(x - a) + \tfrac{1}{2}(\sin x \cos x - \sin a \cos a) \\
&= \tfrac{1}{2}x + \tfrac{1}{2}\sin x \cos x \underbrace{- \tfrac{1}{2}a - \tfrac{1}{2}\sin a \cos a}_{\equiv C} \\
&= \tfrac{1}{2}x + \tfrac{1}{2}\sin x \cos + C\,, \qquad (41)
\end{aligned}$$

so that the primitive function F of $\cos^2 x$ is

$$F(x) = \tfrac{1}{2}x + \tfrac{1}{2}\sin x \cos + C\,. \qquad (42)$$

To verify this result by direct differentiation, we have

$$\begin{aligned}
\frac{dF}{dx} &= \tfrac{1}{2} + \tfrac{1}{2}\cos^2 x - \tfrac{1}{2}\sin^2 x \\
&= \tfrac{1}{2}\cos^2 x + \tfrac{1}{2}(1 - \cos^2 x) \\
&= \cos^2 x\,. \qquad (43)
\end{aligned}$$