First-Year Mathematics

Problem Set 1 January 7, 2005

1. Calculate the derivatives of the following functions from first principles :

   (a) x^3

   (b) $x^{1/2}$

   (c) $x^{-1/2}$

   Ans: (a) $3x^2$, (b) $\frac{1}{2}x^{-1/2}$, (c) $-\frac{1}{2}x^{-3/2}$.

2. Consider the composite function $f(g(x))$, where f and g are differentiable functions. To determine the derivative of $f(g(x))$ with respect to x, begin with the definition

$$\frac{d\,f(g)}{dx} = \lim_{\Delta x \to 0} \left[\frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} \right] .$$

   Multiply and divide this expression by $g(x+\Delta x) - g(x)$ and define the quantity $\Delta g = g(x+\Delta x) - g(x)$. The continuity of g implies that $\Delta g \to 0$ as $\Delta x \to 0$. Hence, obtain the "chain rule":

$$\frac{d\,f(g)}{dx} = \frac{df}{dg}\frac{dg}{dx} . \qquad (1)$$

3. All of the usual rules associated with ordinary differentiation also apply to partial differentiation. Thus, use the chain rule to calculate the partial derivatives of each of the following expressions with respect to the indicated variables.

   (a) $\sqrt{x^2+y^2}$

   (b) $(x^2+y^2+z^2)^{-1/2}$

   (c) $\ln(xy)$

   (d) $e^{x/y}$

   Ans: With ∂_x, ∂_y, and ∂_z signifying partial derivatives with respect to x, y, and z, respectively,

   (a) $\partial_x = x(x^2+y^2)^{-1/2}$, $\partial_y = y(x^2+y^2)^{-1/2}$;

   (b) $\partial_x = -x(x^2+y^2+z^2)^{-3/2}$, $\partial_y = -y(x^2+y^2+z^2)^{-3/2}$, $\partial_z = -z(x^2+y^2+z^2)^{-3/2}$;

   (c) $\partial_x = 1/x$, $\partial_y = 1/y$;

   (d) $\partial_x = y^{-1}\mathrm{e}^{x/y}$, $\partial_y = -xy^{-2}\mathrm{e}^{x/y}$.

4. For small arguments Δx, trigonometric arguments can be used to show that the sine and cosine functions approach the limits

$$\cos(\Delta x) \to 1 \qquad \text{and} \qquad \sin(\Delta x) \to \Delta x\,.$$

Use these expressions in the definition of the derivative, together with the formulas for the sines and cosines of the sum of two angles, to show that

$$\frac{d(\sin x)}{dx} = \cos x \qquad \text{and} \qquad \frac{d(\cos x)}{dx} = -\sin x\,.$$

5. Use the Fundamental Theorem of Calculus to show that reversing the upper and lower limits of integration changes the sign of an integral:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\,.$$

6. Use the Fundamental Theorem of Calculus to deduce that

$$\frac{d}{dx}\left[\int_a^x f(s)\,ds\right] = f(x)$$

and

$$\frac{d}{dx}\left[\int_x^b f(s)\,ds\right] = -f(x)\,.$$

More generally, show that

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(s)\,ds\right] = \frac{dv}{dx}f[v(x)] - \frac{du}{dx}f[u(x)]\,.$$

7. We showed in Classwork 1 that

$$\int_a^b \cos^2 x\,dx = \tfrac{1}{2}(b-a) + \tfrac{1}{2}(\sin b\cos b - \sin a\cos a)\,.$$

Use this integral and the Fundamental Theorem of Calculus to deduce that the primitive function (or anti-derivative) of $\cos^2 x$ is

$$\tfrac{1}{2}x + \tfrac{1}{2}\sin x\cos x + C\,,$$

where C is a constant. Verify this result by direct differentiation.