First-Year Mathematics

Solutions to Classwork 9 March 4, 2005

1. The line integral

$$\oint_{\partial A} (2y\,dx + 3x\,dy)\,, \tag{1}$$

around the boundary of the square whose vertices are at $(1,1)$, $(2,1)$, $(2,2)$, and $(1,2)$ is evaluated by determining the contribution from each straight segment of the boundary. Beginning at $(1,1)$ and proceeding in a counterclockwise direction to $(2,1)$, we have that $y = 1$, $dy = 0$, and $1 \le x \le 2$. Therefore, the contribution to the integral is

$$2\int_1^2 dx = 2\,. \tag{2}$$

Along the segment $(2,1) \to (2,2)$, $x = 2$, $dx = 0$, and $1 \le y \le 2$, so this contribution to the integral is

$$6\int_1^2 dy = 6\,. \tag{3}$$

Along the segment $(2,2) \to (1,2)$, $y = 2$, $dy = 0$, and $1 \le x \le 2$, so this contribution is

$$4\int_2^1 dx = -4\int_1^2 dx = -4\,. \tag{4}$$

Finally, along the segment $(1,2) \to (1,1)$, $x = 1$, $dx = 0$, and $1 \le y \le 2$, and this contribution is

$$3\int_2^1 dy = -3\int_1^2 dy = -3\,. \tag{5}$$

Thus, collecting all of the contributions, we obtain

$$\begin{aligned} \oint_{\partial A} (2y\,dx + 3x\,dy) &= 2\int_1^2 dx + 6\int_1^2 dy - 4\int_1^2 dx - 3\int_1^2 dy \\ &= 2 + 6 - 4 - 3 = 1\,. \end{aligned} \tag{6}$$

2. In the standard notation of Green's theorem, we identify

$$P(x,y) = 2y, \qquad Q(x,y) = 3x\,. \tag{7}$$

Hence,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3 - 2 = 1\,, \tag{8}$$

so the right-hand side of Green's theorem is

$$\int_1^2\int_1^2 dx\,dy = \int_1^2 dx\int_1^2 dy = 1\,, \tag{9}$$

which agrees with the calculation in Part 1.

3. In evaluating the line integral over the boundary of the triangle, we observe that the only new calculation that needs to be carried out is that over the segment $(1,1) \to (2,2)$: the integrals over the segments $(2,2) \to (1,2)$ and $(1,2) \to (1,1)$ are the same as those as in Part 1. Along $(1,1) \to (2,2)$ we have that $y = x$, so $dy = dx$ and x and y both range between 1 and 2. Hence, the line integral over this segment is

$$5\int_1^2 x\,dx = \tfrac{5}{2}x^2\Big|_1^2 = 10 - \tfrac{5}{2} = \tfrac{15}{2}\,. \tag{10}$$

The integral around the boundary of the triangle is therefore given by

$$\begin{aligned}\oint_{\partial A}(2y\,dx + 3x\,dy) &= 5\int_1^2 x\,dx - 4\int_1^2 dx - 3\int_1^2 dy \\ &= \tfrac{15}{2} - 4 - 3 = \tfrac{1}{2}\,. \end{aligned}\tag{11}$$

4. The right-hand side of Green's theorem can be evaluated by writing the integral over the triangle as

$$\begin{aligned}\int_1^2 dy\int_1^y dx &= \int_1^2 (y-1)dy \\ &= \tfrac{1}{2}y^2\Big|_1^2 - y\Big|_1^2 \\ &= 2 - \tfrac{1}{2} - 2 + 1 = \tfrac{1}{2}\,, \end{aligned}\tag{12}$$

which agrees with the calculation in Part 3.

5. For the line integral

$$\oint_{\partial A}(P\,dx + Q\,dy) \tag{13}$$

to vanish for *every* closed path, it is necessary and sufficient that

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}\,. \tag{14}$$

We can also see directly from Green's theorem in the plane that this condition forces the right-hand side to vanish identically. For the line integrals in Parts 1 and 3, we have calculated in Part 2 that

$$\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}\,, \tag{15}$$

so obtaining different results along different closed paths is to be expected.