First-Year Mathematics

Classwork 9 | Green's Theorem | March 4, 2005

Green's theorem states that

$$\oint_{\partial A} (P\,dx + Q\,dy) = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy\,,$$

where $P = P(x, y)$, $Q = Q(x, y)$, A is an area with boundary ∂A, and the line integral is taken in the *counterclockwise* direction.

1. Consider the line integral

   $$\oint_{\partial A} (2y\,dx + 3x\,dy)\,,$$

   where ∂A is the boundary of the square whose vertices are at $(1, 1)$, $(2, 1)$, $(2, 2)$, and $(1, 2)$. Show that, along ∂A, this integral is

   $$2\int_1^2 dx + 6\int_1^2 dy - 4\int_1^2 dx - 3\int_1^2 dy\,.$$

   Thus, evaluate the left-hand side of Green's theorem.

2. Identify the quantities P and Q in the line integral in Part 1 and show that the right-hand side of Green's theorem reduces to

   $$\int_1^2 dx \int_1^2 dy$$

   and verify that this agrees with the result of Part 1.

3. Evaluate the line integral in Part 1 over the boundary of the triangle whose vertices are at $(1, 1)$, $(2, 2)$, and $(1, 2)$.

   Answer: $\frac{1}{2}$.

4. Evaluate the right-hand side of Green's theorem over the interior of this triangle and verify that your result agrees with that obtained in Part 3.

5. The evaluation of the line integrals in Parts 1 and 3 yielded different results. How could this have been anticipated?