First-Year Mathematics

Solutions to Classwork 8 — The Divergence Theorem — February 25, 2005

1. (a) The divergence of $\boldsymbol{V} = 3x^2y\,\boldsymbol{i} - 2xy^2\,\boldsymbol{j} - xyz\,\boldsymbol{k}$ is

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = 6xy - 4xy - xy = xy\,. \tag{1}$$

(b) The integral of this quantity over the cube is

$$\begin{aligned}\iiint \boldsymbol{\nabla}\cdot\boldsymbol{V}\,d\tau &= \int_0^a dx\int_0^a dy\int_0^a dz\,xy\\ &= \underbrace{\int_0^a x\,dx}_{\frac{1}{2}a^2}\underbrace{\int_0^a y\,dy}_{\frac{1}{2}a^2}\underbrace{\int_0^a dz}_{a} = \tfrac{1}{4}a^5\,.\end{aligned} \tag{2}$$

2. (a) The face with vertices at $(0,0,0)$, $(0,a,0)$, $(0,0,a)$, and $(0,a,a)$. This face lies within the y–z plane, so

$$\boldsymbol{n} = -\boldsymbol{i}\,. \tag{3}$$

The "dot" product $\boldsymbol{V}\cdot\boldsymbol{n}$ is

$$\boldsymbol{V}\cdot\boldsymbol{n} = -3x^2y\,. \tag{4}$$

On this face, $x=0$, so

$$\boldsymbol{V}\cdot\boldsymbol{n} = 0\,. \tag{5}$$

Therefore,

$$\int_0^a dy\int_0^a dz\,\boldsymbol{V}\cdot\boldsymbol{n} = 0\,. \tag{6}$$

(b) The face with vertices at $(a,0,0)$, $(a,a,0)$, $(a,0,a)$, and (a,a,a). This face lies parallel to the y–z plane and intersects the x-axis at $x=a$. Thus,

$$\boldsymbol{n} = \boldsymbol{i}\,. \tag{7}$$

The "dot" product $\boldsymbol{V}\cdot\boldsymbol{n}$ is

$$\boldsymbol{V}\cdot\boldsymbol{n} = 3x^2y\,. \tag{8}$$

On this face, $x=a$, so

$$\boldsymbol{V}\cdot\boldsymbol{n} = 3a^2y\,. \tag{9}$$

Therefore,

$$\int_0^a dy\int_0^a dz\,3a^2y = 3a^2\underbrace{\int_0^a y\,dy}_{\frac{1}{2}a^2}\underbrace{\int_0^a dz}_{a} = \tfrac{3}{2}a^5\,. \tag{10}$$

(c) The face with vertices at $(0,0,0)$, $(a,0,0)$, $(0,0,a)$, and $(a,0,a)$. This face lies within the x–z plane, so

$$\boldsymbol{n} = -\boldsymbol{j}\,. \tag{11}$$

The "dot" product $\boldsymbol{V}\cdot\boldsymbol{n}$ is

$$\boldsymbol{V}\cdot\boldsymbol{n} = 2xy^2\,. \tag{12}$$

On this face, $y = 0$, so

$$\boldsymbol{V}\cdot\boldsymbol{n} = 0\,. \tag{13}$$

Therefore,

$$\int_0^a dx \int_0^a dz\, \boldsymbol{V}\cdot\boldsymbol{n} = 0\,. \tag{14}$$

(d) The face with vertices at $(0,a,0)$, $(a,a,0)$, $(0,a,a)$, and (a,a,a). This face lies parallel to the x–z plane, intersecting the y-axis at $y = a$. Thus,

$$\boldsymbol{n} = \boldsymbol{j}\,. \tag{15}$$

The "dot" product $\boldsymbol{V}\cdot\boldsymbol{n}$ is

$$\boldsymbol{V}\cdot\boldsymbol{n} = -2xy^2\,. \tag{16}$$

On this face, $y = a$, so

$$\boldsymbol{V}\cdot\boldsymbol{n} = -2xa^2\,. \tag{17}$$

Therefore,

$$-\int_0^a dx \int_0^a dz\, 2xa^2 = -2a^2 \underbrace{\int_0^a x\,dx}_{\frac{1}{2}a^2} \underbrace{\int_0^a dz}_{a} = -a^5\,. \tag{18}$$

(e) The face with vertices at $(0,0,0)$, $(a,0,0)$, $(0,a,0)$, and $(a,a,0)$. This face lies within the x–y plane, so

$$\boldsymbol{n} = -\boldsymbol{k}\,. \tag{19}$$

The "dot" product $\boldsymbol{V}\cdot\boldsymbol{n}$ is

$$\boldsymbol{V}\cdot\boldsymbol{n} = xyz\,. \tag{20}$$

On this face, $z = 0$, so

$$\boldsymbol{V}\cdot\boldsymbol{n} = 0\,. \tag{21}$$

Therefore,

$$\int_0^a dx \int_0^a dy\, \boldsymbol{V}\cdot\boldsymbol{n} = 0\,. \tag{22}$$

(f) The face with vertices at $(0,0,a)$, $(a,0,a)$, $(0,a,a)$, and (a,a,a). This face lies parallel to the x–y plane, intersecting the z-axis at $z=a$. Thus,

$$\boldsymbol{n} = \boldsymbol{k}\,. \tag{23}$$

The "dot" product $\boldsymbol{V}\cdot\boldsymbol{n}$ is

$$\boldsymbol{V}\cdot\boldsymbol{n} = -xyz\,. \tag{24}$$

On this face, $z=a$, so

$$\boldsymbol{V}\cdot\boldsymbol{n} = -xya\,. \tag{25}$$

Therefore,

$$-\int_0^a dx \int_0^a dy\, xya = -\underbrace{\int_0^a x\,dx}_{\frac{1}{2}a^2}\underbrace{\int_0^a y\,dy}_{\frac{1}{2}a^2} = -\tfrac{1}{4}a^5\,. \tag{26}$$

3. Summing the contributions to the surface integral from each face, we obtain

$$\iint \boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma = 0 + \tfrac{3}{2}a^5 + 0 - a^5 + 0 - \tfrac{1}{4}a^5 = \tfrac{1}{4}a^5\,, \tag{27}$$

which agrees with the integration of the divergence of $\boldsymbol{V}$ in Part 1(a).