First-Year Mathematics

Classwork 8 | The Divergence Theorem | February 25, 2005

The Divergence Theorem in three dimensions stipulates that the divergence of a vector field, integrated over a volume, is equal to the flux of the vector field across the boundary of that volume:

$$\iiint \boldsymbol{\nabla} \cdot \boldsymbol{V} \, d\tau = \iint \boldsymbol{V} \cdot \boldsymbol{n} \, d\sigma \, .$$

In this Classwork, you will verify this theorem by evaluating these quantities explicitly for a particular vector field and volume.

1. Consider the vector field

$$\boldsymbol{V} = 3x^2 y \, \boldsymbol{i} - 2xy^2 \, \boldsymbol{j} - xyz \, \boldsymbol{k} \, .$$

   (a) Determine the divergence of this vector field.

   Answer: $\boldsymbol{\nabla} \cdot \boldsymbol{V} = xy$.

   (b) Evaluate the triple integral of this quantity over the interior of the cube of side a in the first octant whose base has the vertices $(0,0,0)$, $(a,0,0)$, $(0,a,0)$, and $(a,a,0)$.

   Answer: $\frac{1}{4}a^5$.

2. The evaluation of the right-hand side of the Divergence Theorem above requires integrating the normal component of $\boldsymbol{V}$ over each face of the cube. For each face of the cube, proceed as follows:

   (a) Identify the *outward* unit normal $\boldsymbol{n}$, i.e. the direction of the unit vector that points away from the interior of the cube.

   (b) Calculate the "dot" product $\boldsymbol{V} \cdot \boldsymbol{n}$ to find the component of $\boldsymbol{V}$ along $\boldsymbol{n}$.

   (c) For each face of the cube, one of the variables x, y, or z is a constant. Use this to determine the appropriate form of the quantity determined in (b).

   (d) Evaluate the double integral of the quantity determined in (c) over the side of the cube.

3. Add the contributions calculated for each of the six faces in Part 2 and show that the sum equals the quantity calculated in Part 1(a).