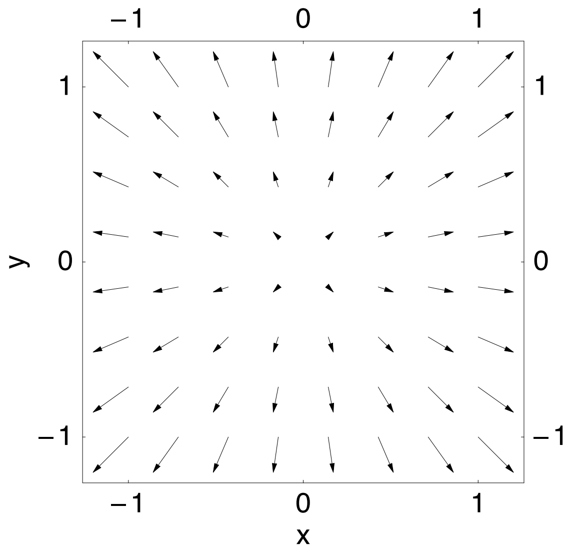
First-Year Mathematics

Solutions to Classwork 7 The Meaning of the Divergence February 18, 2005

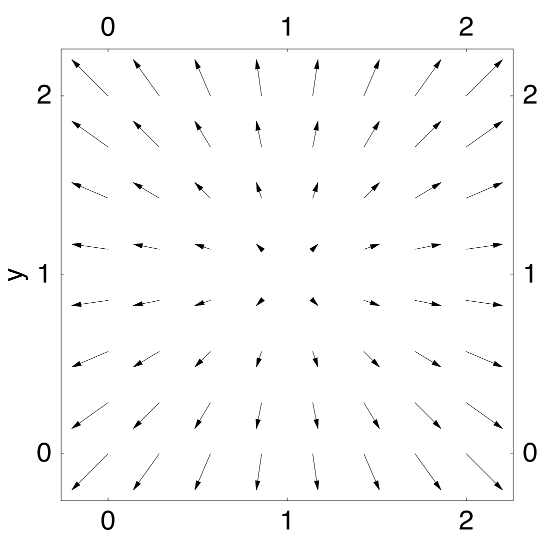
The vector field $\boldsymbol{V} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$ is plotted below in the region $-1 \le x \le 1$ and $-1 \le y \le 1$:



1. The relative velocity field $\boldsymbol{V}(x,y) - \boldsymbol{V}(x_0,y_0) = (x - x_0)\,\boldsymbol{i} + (y - y_0)\,\boldsymbol{j}$ represents a shift of the origin of $\boldsymbol{V}$. Its divergence is

$$\boldsymbol{\nabla} \cdot [(x - x_0)\,\boldsymbol{i} + (y - y_0)\,\boldsymbol{j}] = 2\,,$$

as before. The relative velocity field is plotted below for $x_0 = 1$ and $y_0 = 1$ in the region $0 \le x \le 2$ and $0 \le y \le 2$:

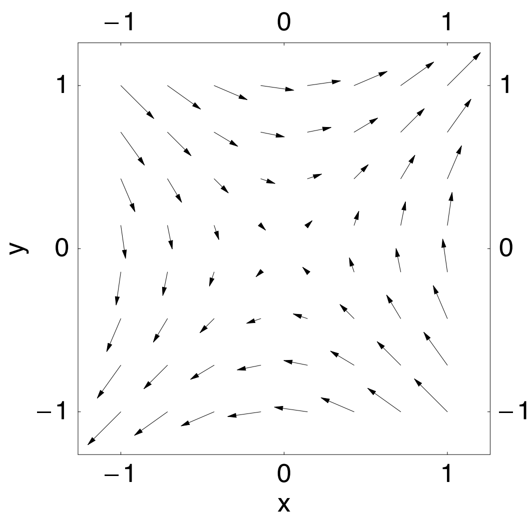


Note that this field is *identical* to the original vector field. The same result would have been obtained had we chosen *any* other point because the divergence is independent of position. The fact that the divergence is positive is evident from the vectors pointing away from the central point, just as in the original diagram. In the fluid analogy, all of the particles are moving away from our particle.

2. The divergence of $\boldsymbol{V}(x,y) = (ax+by)\,\boldsymbol{i} + (cx+dy)\,\boldsymbol{j}$ is

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = a + d\,.$$

Thus, the divergence is positive if $a+d>0$, negative if $a+d<0$, and vanishes if $a+d=0$ (including the possibility that $a=0$ and $d=0$). Notice that these conclusions are valid regardless of the values of b and c because these constants are eliminated upon taking the derivatives to form the divergence. The vector field $\boldsymbol{V} = y\,\boldsymbol{i} + x\,\boldsymbol{j}$, which has zero divergence everywhere, is shown below in the region $-1 \le x \le 1$ and $-1 \le y \le 1$:



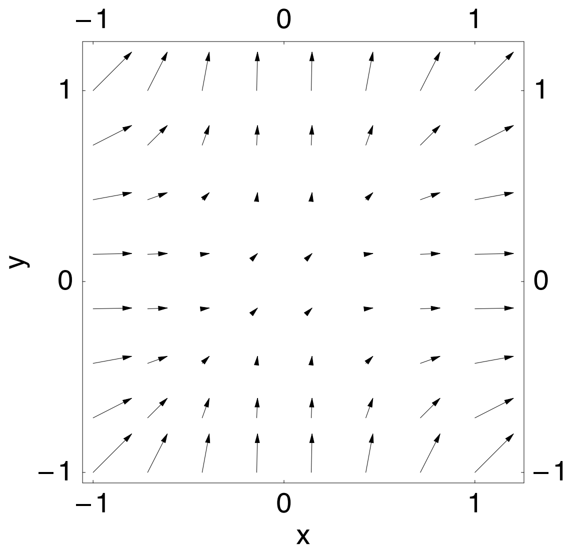
The interpretation of a vector field having zero divergence in terms of the fluid model in Part 1 is that the convergence of particles toward our particle is exactly balanced by those diverging from our particle.

3. The divergence of $\boldsymbol{V}(x,y) = x^2\,\boldsymbol{i} + y^2\,\boldsymbol{j}$ is

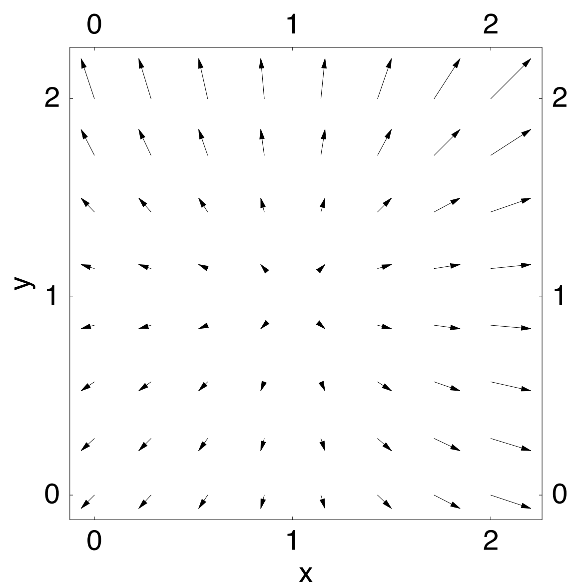
$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = 2x + 2y\,,$$

which is now position-dependent and can take positive values, negative values, or vanish. The divergence vanishes along the line $y=-x$, which includes the origin.

This vector field is plotted below in the region $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$:

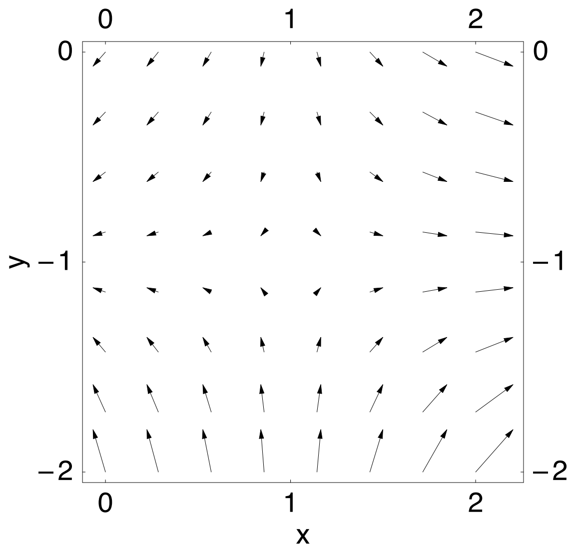


In the fluid analogy of Part 1, the vanishing divergence is seen to result from the convergence and divergence of particles exactly balancing, as in Part 2. The vector field relative to the point $(1, 1)$ is shown below in the region $0 \leq x \leq 2$ and $0 \leq y \leq 2$:

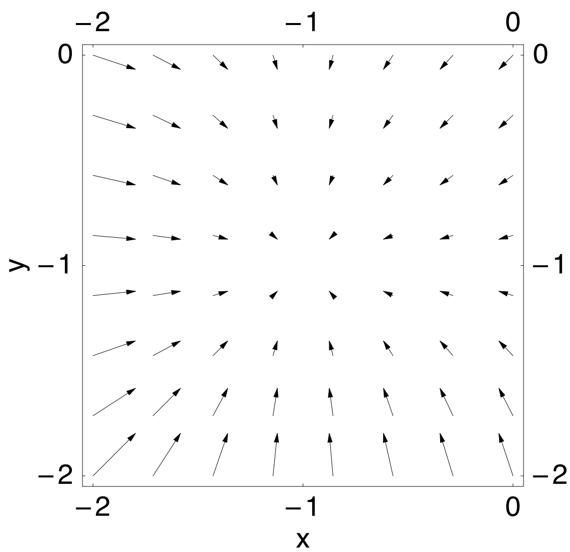


The positive divergence results from the dominance of particles diverging from our particle.

The vector field relative to the point $(1, -1)$ is shown below in the region $0 \leq x \leq 2$ and $-2 \leq y \leq 0$:



The vanishing divergence is again seen to result from the convergence and divergence of particles exactly balancing. The vector field relative to the point $(-1, -1)$ is shown below in the region $-2 \leq x \leq 0$ and $-2 \leq y \leq 0$:



The negative divergence results from the dominance of particles converging on our particle.