First-Year Mathematics

Classwork 7 | The Meaning of the Divergence | February 18, 2005

Consider the vector field

$$\boldsymbol{V}(x,y) = x\,\boldsymbol{i} + y\,\boldsymbol{j}\,.$$

We showed in lectures that the divergence of this vector field is given by

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = 2\,.$$

Sketch this vector field. You should see radial vectors pointing away from the origin showing what appears to be an outward flux from the origin. However, $\boldsymbol{\nabla}\cdot\boldsymbol{V} = 2$ at *all* points. Understanding this is the subject of this Classwork.

1. Imagine that $\boldsymbol{V}$ is the velocity field of a fluid and that you are riding on a fluid particle. You will notice the relative velocity of particles, i.e. whether you are on a collision course with them (*con*verging), or being separated from them (*di*verging), or neither. The relative velocities are computed by subtracting the instantaneous velocity at your point $\boldsymbol{V}(x_0,y_0)$ from the instantaneous velocity at all other points. This gives the relative velocity field

   $$\boldsymbol{V}(x,y) - \boldsymbol{V}(x_0,y_0)\,.$$

   Compute the divergence of this vector field. Choose a reference point, say $(1,1)$, and sketch the relative velocity field and compare with your sketch of $\boldsymbol{V}$. What can you conclude about the divergence of $\boldsymbol{V}$ having the same value everywhere?

2. Consider the vector field

   $$\boldsymbol{V}(x,y) = (ax+by)\,\boldsymbol{i} + (cx+dy)\,\boldsymbol{j}\,,$$

   where a, b, c, and d are constants. Determine the conditions that determine whether the divergence is positive, negative, or zero. As an example of a vector field with zero divergence, consider

   $$\boldsymbol{V}(x,y) = y\,\boldsymbol{i} + x\,\boldsymbol{j}\,.$$

   Provide an interpretation for this vector field having zero divergence using the analogy in Part 1.

3. Consider the vector field

   $$\boldsymbol{V}(x,y) = x^2\,\boldsymbol{i} + y^2\,\boldsymbol{j}\,.$$

   Follow the steps in Part 1 for the points $(1,1)$, $(1,-1)$, and $(-1,-1)$ and provide an interpretation of your results.