First-Year Mathematics

Solutions to Classwork 6   Directional Derivatives and the Gradient   February 11, 2005

1. (a) The lines of constant $f(x,y) = y$ are parallel to the x-axis. The gradient of a function $f(x,y)$ is

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}\,, \tag{1}$$

so the gradient of $f(x,y) = y$ is

$$\boldsymbol{\nabla} f = \boldsymbol{j}\,, \tag{2}$$

which is the unit vector in the y-direction, i.e. perpendicular to the lines of constant f.

(b) The lines of constant $g(x,y) = y^2$ are also parallel to the x-axis. The gradient of this function is

$$\boldsymbol{\nabla} g = 2y\,\boldsymbol{j}\,, \tag{3}$$

which is also parallel to the y-axis, but the magnitude $2y$ reflects the fact that lines of constant g increase quadratically, rather than linearly, as is the case for f.

2. The surfaces of constant $f(x,y,z) = z$ are planes parallel to the x-y plane. The gradient of a function $f(x,y,z)$ is

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}\,, \tag{4}$$

so the gradient of $f(x,y,z) = z$ is

$$\boldsymbol{\nabla} f = \boldsymbol{k}\,, \tag{5}$$

which is the unit vector along the z-direction, i.e. perpendicular to the planes of constant f.

3. The gradient of $f(x,y,z) = xy$ is

$$\boldsymbol{\nabla} f = y\,\boldsymbol{i} + x\,\boldsymbol{j}\,. \tag{6}$$

The direction $\boldsymbol{u}$ is given as $\boldsymbol{u} = 3\,\boldsymbol{i} + 4\,\boldsymbol{j}$. The unit vector along this direction is obtained by dividing this vector by its magnitude:

$$\frac{\boldsymbol{u}}{|\boldsymbol{u}|} = \tfrac{3}{5}\,\boldsymbol{i} + \tfrac{4}{5}\,\boldsymbol{j}\,. \tag{7}$$

Hence, the directional derivative of xy along the direction of $\boldsymbol{u}$ at (1,1) is

$$(\boldsymbol{i} + \boldsymbol{j}) \cdot (\tfrac{3}{5}\,\boldsymbol{i} + \tfrac{4}{5}\,\boldsymbol{j}) = \tfrac{7}{5}\,. \tag{8}$$

4. The gradient of $f(x,y,z) = x^2 + y^2 - z^2$ is

$$\boldsymbol{\nabla} f = 2x\,\boldsymbol{i} + 2y\,\boldsymbol{j} - 2z\,\boldsymbol{k}\,. \tag{9}$$

The unit vector along $\boldsymbol{i} + 2\,\boldsymbol{j} - 2\,\boldsymbol{k}$ is

$$\frac{\boldsymbol{u}}{|\boldsymbol{u}|} = \tfrac{1}{3}\,\boldsymbol{i} + \tfrac{2}{3}\,\boldsymbol{j} - \tfrac{2}{3}\,\boldsymbol{k}\,. \tag{10}$$

Hence, the derivative of f along this direction at the point $(1,1,2)$ is

$$(2\,\boldsymbol{i} + 2\,\boldsymbol{j} - 4\,\boldsymbol{k}) \cdot (\tfrac{1}{3}\,\boldsymbol{i} + \tfrac{2}{3}\,\boldsymbol{j} - \tfrac{2}{3}\,\boldsymbol{k}) = \tfrac{14}{3}\,. \tag{11}$$

5. The gradient of $f(x,y,z) = x^3 - xyz + z^3$ is

$$\boldsymbol{\nabla} f = (3x^2 - yz)\,\boldsymbol{i} - xz\,\boldsymbol{j} + (3z^2 - xy)\,\boldsymbol{k}\,. \tag{12}$$

At the point $(1,1,1)$, this reduces to

$$2\,\boldsymbol{i} - \boldsymbol{j} + 2\,\boldsymbol{k} \tag{13}$$

Since this vector is normal to surfaces of constant f, and the point $(1,1,1)$ lies on the surface

$$x^3 - xyz + z^3 = 1\,, \tag{14}$$

the *unit* vector $\boldsymbol{n}$ which points normal to this surface at the given point is

$$\boldsymbol{n} = \frac{\boldsymbol{\nabla} f}{|\boldsymbol{\nabla} f|} = \tfrac{1}{3}(2\,\boldsymbol{i} - \boldsymbol{j} + 2\,\boldsymbol{k})\,. \tag{15}$$

6. The gradient of $T(x,y) = x^2 - y^2$ is

$$\boldsymbol{\nabla} T = 2x\,\boldsymbol{i} - 2y\,\boldsymbol{j}\,. \tag{16}$$

The direction of heat flow $\boldsymbol{J}$ is $-\boldsymbol{\nabla} T$:

$$\boldsymbol{J}(x,y) = -2x\,\boldsymbol{i} + 2y\,\boldsymbol{j}\,. \tag{17}$$

Thus, the heat flows at the given points (x,y) are:

(a) At the origin,

$$\boldsymbol{J}(0,0) = \boldsymbol{0}\,. \tag{18}$$

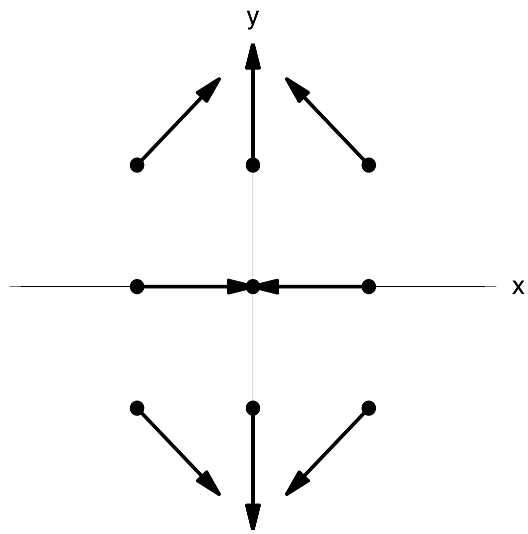
(b) At $(1,0)$, $(-1,0)$, $(0,1)$, and $(0,1)$,

$$\boldsymbol{J}(1,0) = -2\,\boldsymbol{i}, \quad \boldsymbol{J}(-1,0) = 2\,\boldsymbol{i}, \quad \boldsymbol{J}(0,1) = 2\,\boldsymbol{j}, \quad \boldsymbol{J}(0,-1) = -2\,\boldsymbol{j}\,. \tag{19}$$

(c) At $(1,1)$, $(-1,1)$, $(1,-1)$, and $(-1,-1)$,

$$\begin{aligned} \boldsymbol{J}(1,1) &= -2\,\boldsymbol{i} + 2\,\boldsymbol{j}, & \boldsymbol{J}(-1,1) &= 2\,\boldsymbol{i} + 2\,\boldsymbol{j} \\ \boldsymbol{J}(1,-1) &= -2\,\boldsymbol{i} - 2\,\boldsymbol{j}, & \boldsymbol{J}(-1,-1) &= 2\,\boldsymbol{i} - 2\,\boldsymbol{j} \end{aligned} \tag{20}$$

The unit vectors corresponding to the directions are shown below:



The surface $T(x,y)$ with $A = 1$ is: