First-Year Mathematics

Classwork 6 | Directional Derivatives and the Gradient | February 11, 2005

1. (a) Consider the function
$$f(x,y) = y\,.$$
What are the lines of constant f? Determine the gradient of f (in two dimensions) and show that its direction is perpendicular to these lines.

   (b) Now consider the function
$$g(x,y) = y^2\,.$$
Determine the gradient of g and identify the origin of any differences between $\boldsymbol{\nabla} f$ and $\boldsymbol{\nabla} g$.

2. Consider the function
$$f(x,y,z) = z\,.$$
What are the surfaces of constant f? Determine the gradient of f and show that its direction is normal to these surfaces.

3. Find the derivative of $f(x,y) = xy$ at $(1,1)$ in the direction $3\,\boldsymbol{i} + 4\,\boldsymbol{j}$.

4. Find the derivative of $f(x,y,z) = x^2 + y^2 - z^2$ at $(1,1,2)$ in the direction $\boldsymbol{i} + 2\,\boldsymbol{j} - 2\,\boldsymbol{k}$.

5. Determine a unit vector normal to the surface
$$x^3 - xyz + z^3 = 1$$
at the point $(1,1,1)$.

6. Suppose that the temperature within a region is described by the function $T(x,y) = A + x^2 - y^2$, where A is a constant. Given that heat flows in the direction *opposite* to the temperature gradient (i.e. from high temperature to low temperature), sketch the direction of heat flow at the following points:

   (a) The origin.

   (b) $(1,0)$, $(-1,0)$, $(0,1)$, $(0,-1)$.

   (c) $(1,1)$, $(-1,1)$, $(1,-1)$, $(-1,-1)$.

   Sketch $T(x,y)$ in a neighborhood of the origin from these heat flows.