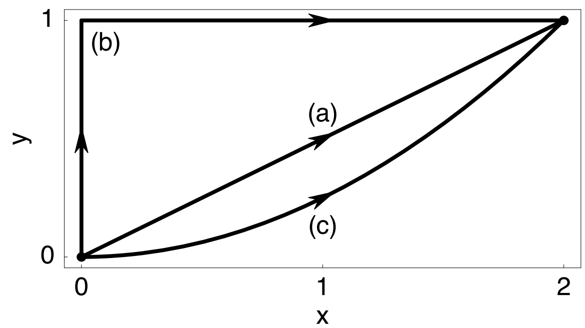
First-Year Mathematics

Solutions to Classwork 5 February 4, 2005

1. The integration paths are shown below:



(a) To evaluate the integral I_1 over x, we use the relation $y = \frac{1}{2}x$ along the path to write the integrand as

$$f(x, y) = \tfrac{1}{2}x^2 + \tfrac{1}{2}x\,.$$

The line integral from $(0, 0)$ to $(2, 1)$ is then carried out as an ordinary integral:

$$\int_0^2 (\tfrac{1}{2}x^2 + \tfrac{1}{2}x)\,dx = \tfrac{1}{6}x^3\Big|_0^2 + \tfrac{1}{4}x^2\Big|_0^2 = \tfrac{4}{3} + 1 = \tfrac{7}{3}\,.$$

To evaluate the integral I_2 over y, we use $x = 2y$ along the path to write the integrand as

$$f(x, y) = 2y^2 + y\,,$$

so the integral over y is

$$\int_0^1 (2y^2 + y)\,dy = \tfrac{2}{3}y^3\Big|_0^1 + \tfrac{1}{2}y^2\Big|_0^1 = \tfrac{2}{3} + \tfrac{1}{2} = \tfrac{7}{6}\,.$$

(b) The integral I_1 over x vanishes along the line $x = 0$ (because $dx = 0$). Along $y = 1$ (from $x = 0$ to $x = 2$), this integral is

$$\int_0^2 (x + 1)\,dx = \tfrac{1}{2}x^2\Big|_0^2 + x\Big|_0^2 = 2 + 2 = 4\,.$$

For the integral I_2 over y, along $x = 0$, $f(x, y) = y$ so the integral over this segment is

$$\int_0^1 y\,dy = \tfrac{1}{2}y^2\Big|_0^1 = \tfrac{1}{2}\,.$$

For the integral along $y = 1$, $dy = 0$, so the integral along this segment vanishes.

(c) For the integral I_1 over x, we have

$$f[x(t), y(t)] = 2t^3 + t^2$$

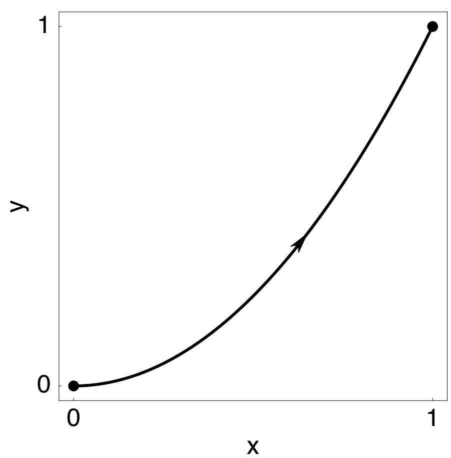
along the path and $dx = 2\,dt$. Thus, this integral becomes

$$\int_0^1 (4t^3 + 2t^2)\,dt = t^4\Big|_0^1 + \tfrac{2}{3}t^3\Big|_0^1 = 1 + \tfrac{2}{3} = \tfrac{5}{3}\,.$$

For the integral I_2 over y, we have $dy = 2t\,dt$ along the path, so this integral is

$$\int_0^1 (4t^4 + 2t^3)\,dt = \tfrac{4}{5}t^5\Big|_0^1 + \tfrac{1}{2}t^4\Big|_0^1 = \tfrac{4}{5} + \tfrac{1}{2} = \tfrac{13}{10}\,.$$

2. The integration path is shown below:



Along the line $y = x^2$, the function $f(x, y) = x^2 + y$ becomes

$$f(x, y) = x^2 + x^2 = 2x^2\,.$$

Thus, the integral is

$$2\int_0^1 x^2\,dx = \tfrac{2}{3}x^3\Big|_0^1 = \tfrac{2}{3}\,.$$

But we can also carry out this integral over y, because along this curve $f(x, y)$ can be written as

$$f(x, y) = y + y = 2y\,.$$

Along $y = x^2$, we have that $dy = 2x\,dx$, or that

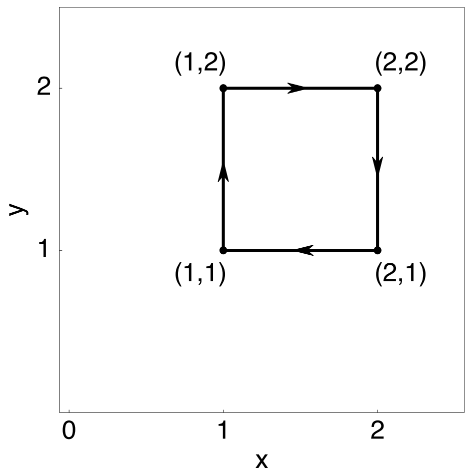
$$dx = \frac{dy}{2x} = \frac{dy}{2\sqrt{y}}\,.$$

Thus, the integral over y:

$$\int_0^1 \sqrt{y}\,dy = \tfrac{2}{3}y^{3/2}\Big|_0^1 = \tfrac{2}{3}\,,$$

as before.

3. The integration path is shown below:



Along the segments where x is constant, i.e. $(1,1) \to (1,2)$ and $(2,2) \to (2,1)$, $dx = 0$, so these contributions to the integral vanish. Along the segment $(1,2) \to (2,2)$, $y = 2$ and the contribution to the integral is

$$2\int_1^2 dx = 2\,.$$

Along the segment $(2,1) \to (1,1)$, $y = 1$ and the contribution to the integral is

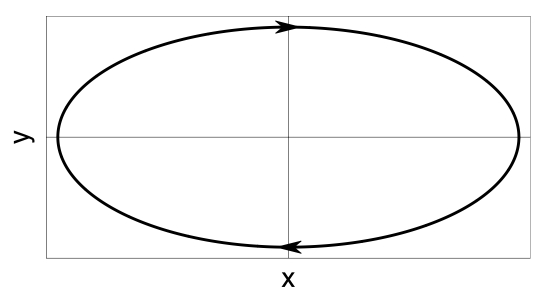
$$\int_2^1 dx = -1\,.$$

Thus, for the prescribed path,

$$\oint y\,dx = 2 - 1 = 1\,,$$

which is equal to the area enclosed by this path (i.e. a square with sides of unit length).

4. The closed curve is parametrized by $x = a\cos t$ and $y = -b\sin t$ for $0 \le t < 2\pi$, which traces out a path in the *clockwise* direction, as shown below for the case $a > b$:



The values of x and y so parametrized satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,,$$

which is a circle if $a = b$ and an ellipse if $a \neq b$. To integrate $y\,dx$ over this curve, we convert it into an integral over t by using

$$y = -b\sin t, \qquad dx = -a\sin t\,dt\,.$$

Thus, the integral becomes

$$\oint y\,dx = ab\int_0^{2\pi} \sin^2 t\,dt = \pi ab\,,$$

which is the area of an ellipse ($a \neq b$) or a circle ($a = b$) (cf. Part 5, Classwork 2).