First-Year Mathematics

Classwork 4 Cylindrical and Spherical Polar Coordinates January 28, 2005

1. The procedure described in lectures for deriving the volume element in cylindrical coordinates can be applied to other coordinate systems. Consider the case of spherical polar coordinates,

$$x = r\cos\phi\sin\theta\,, \qquad y = r\sin\phi\sin\theta\,, \qquad z = r\cos\theta\,,$$

where $0 \le r < \infty$, $0 \le \phi < 2\pi$, and $0 \le \theta \le \frac{1}{2}\pi$. Any point (x, y, z) can be expressed as

$$\boldsymbol{r} = r\cos\phi\sin\theta\,\boldsymbol{i} + r\sin\phi\sin\theta\,\boldsymbol{j} + r\cos\theta\,\boldsymbol{k}\,.$$

Take the differential of $\boldsymbol{r}$ in turn with respect to each of the three independent variables to obtain three vectors. Verify that these are mutually orthogonal. Then use the triple scalar product to calculate the volume within the parallelepiped defined by the vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$,

$$\boldsymbol{A}\cdot\boldsymbol{B}\times\boldsymbol{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}\,,$$

where the subscripts indicate the Cartesian components of the corresponding vector, and thereby obtain

$$dV = r^2\sin\theta\,dr\,d\theta\,d\phi\,.$$

Verify that the same result is obtained by taking the product of the lengths of the three differential vectors:

$$dV = |d\boldsymbol{r}_r||d\boldsymbol{r}_\phi||d\boldsymbol{r}_\theta| = r^2\sin\theta\,dr\,d\theta\,d\phi\,. \tag{1}$$

2. Consider a right circular cylinder of radius R and height H. The axis of the cylinder coincides with the z-axis and its base is the plane $z = 0$. Suppose that the mass density ϱ of material within this cylinder varies directly with the height above the base: $\varrho(z) = az$, where a is a constant.

   (a) Use cylindrical coordinates to determine the mass contained within the cylinder.

   (b) Show that the average mass density m in the cylinder is

$$m = \frac{aH}{2}\,.$$

3. Suppose that the volume between the two spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ in the upper half-plane ($z \geq 0$) is filled with a material with a mass density ϱ that varies with the distance from the origin: $\varrho(r) = ar$, where a is a constant.

   (a) Use spherical polar coordinates to determine the mass within this volume.

   (b) Show that the average mass density m in the volume is

$$m = \frac{45b}{28}\,.$$

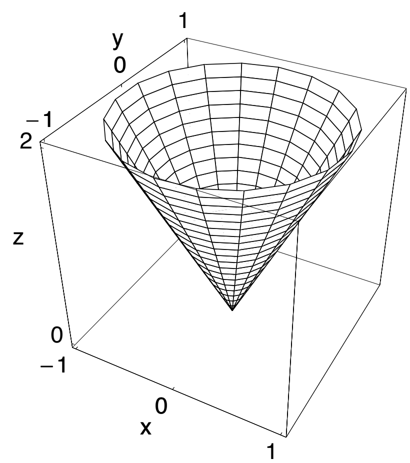
4. Use spherical polar coordinates to find the volume contained within the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,,$$

   where a, b, and c are constants.

   Hint: Transform to a set of coordinates (x', y', z') where the equation of the ellipsoid reduces to $x'^2 + y'^2 + z'^2 = 1$.

5. Consider the volume of a cone given by the equation $x^2+y^2 = \frac{1}{4}z^2$ bounded from above by the plane $z = 2$, as shown below:



   (a) Use cylindrical polar coordinates (r, ϕ, z) to represent this volume as

$$V = \int_0^2 dz \int_0^{2\pi} d\phi \int_0^{\frac{1}{2}z} r\,dr$$

   and evaluate the integral.

   (b) (**Optional**) Use spherical polar coordinates to represent this volume as

$$V = \int_0^{2\pi} d\phi \int_0^{\cos^{-1}(2/\sqrt{5})} \sin\theta\,d\theta \int_0^{2/\cos\theta} r^2\,dr$$

   and evaluate the integral.

   Hint: The equation of the plane $z = 2$ is, in spherical polar coordinates, $r\cos\theta = 2$.