First-Year Mathematics

Solutions to Classwork 3 — Double and Triple Integrals — January 21, 2005

1. The range of x is given as $0 \le x \le 1$. If we take the range of y to be $0 \le y \le \frac{1}{2}$, then the range of z is bounded from below by y and from above by $\frac{1}{2}$. Hence, the volume integral over this region is

$$\iiint_V dx\,dy\,dz = \int_0^1 dx \int_0^{1/2} dy \int_y^{1/2} dz\,.$$

Carrying out each of the one-dimensional integrals yields

$$\int_0^1 dx \int_0^{1/2} dy \int_y^{1/2} dz = \underbrace{\int_0^1 dx}_{1} \int_0^{1/2} dy \left(z \Big|_y^{1/2} \right) = \int_0^{1/2} \left(\tfrac{1}{2} - y\right) dy$$

$$= \frac{y}{2}\Big|_0^{1/2} - \frac{y^2}{2}\Big|_0^{1/2}$$

$$= \frac{1}{4} - \frac{1}{8} = \frac{1}{8}\,.$$

2. The radius r at height z of the cone is a linear function of z:

$$r(z) = A + Bz\,.$$

By requiring that $r(0) = R$, we obtain $A = R$, and by requiring that $r(h) = 0$, we obtain $B = -R/h$. Thus,

$$r(z) = \frac{R}{h}(h - z)\,.$$

The volume of the cone can be thought of as composed of incremental volumes dV given by

$$dV = \pi r^2(z)\,dz\,,$$

which are the areas of circles of radius $r(z)$ multiplied by their "thickness" dz. Hence, the volume of the cone can be calculated as

$$\iiint_V dx\,dy\,dz = \pi \int_0^h r^2(z)\,dz\,.$$

By expanding the factor $r^2(z)$ and carrying the integration over z, we obtain

$$\begin{aligned}
\pi \int_0^h r^2(z)\, dz &= \frac{\pi R^2}{h^2} \int_0^h (h^2 - 2hz + z^2)\, dz \\
&= \frac{\pi R^2}{h^2} \left\{ h^2 z \Big|_0^h - hz^2 \Big|_0^h + \tfrac{1}{3} z^3 \Big|_0^h \right\} \\
&= \frac{\pi R^2}{h^2} \left(h^3 - h^3 + \tfrac{1}{3} h^3 \right) \\
&= \frac{1}{3} \pi R^2 h \,.
\end{aligned}$$

3. (a) To determine the area of the shaded region by using circular polar coordinates (r, ϕ), we need to determine the ranges of r and ϕ for any point within this region. From the coordinates of A, B, and C, the range of ϕ is seen to be

$$0 \le \phi \le \tfrac{1}{4}\pi \,.$$

The range of r cannot be determined independently because the upper bound of the integration region is given by $r = 2\cos(2\phi)$. Thus, for a given value of ϕ, the range of r within the shaded region is therefore given by

$$0 \le r \le 2\cos(2\phi) \,.$$

(b) Since the shaded region corresponds to $\frac{1}{8}$th of the area A of the clover leaf, the ranges of the variables obtained in (a) allow us to write A as the integral

$$A = 8 \int_0^{\frac{1}{4}\pi} d\phi \int_0^{2\cos(2\phi)} r\, dr \,.$$

(c) The evaluation of A is as follows:

$$\begin{aligned}
A &= 8 \int_0^{\frac{1}{4}\pi} d\phi \int_0^{2\cos(2\phi)} r\, dr \\
&= 8 \int_0^{\frac{1}{4}\pi} d\phi \left\{ \frac{r^2}{2} \Bigg|_0^{2\cos(2\phi)} \right\} \\
&= 16 \int_0^{\frac{1}{4}\pi} d\phi \cos^2(2\phi) \qquad (t = \tfrac{1}{2}\phi) \\
&= 8 \underbrace{\int_0^{\frac{1}{2}\pi} dt \cos^2 t}_{\frac{1}{4}\pi} \\
&= 2\pi \,.
\end{aligned}$$