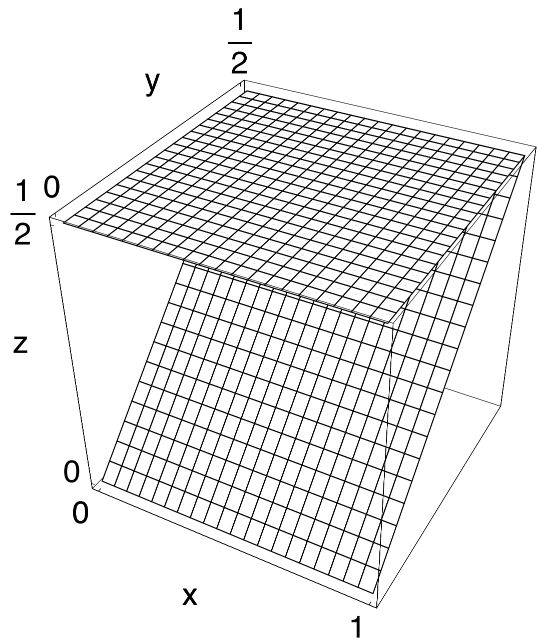
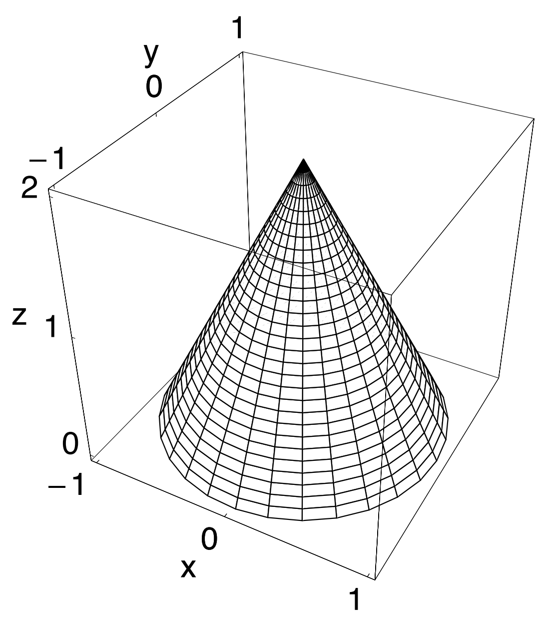
First-Year Mathematics

Classwork 3 | Double and Triple Integrals | January 21, 2005

1. Compute the volume of the region for $0 \leq x \leq 1$ that is bounded from below by the plane $z = y$ and from above by the plane $z = \frac{1}{2}$. This region is shown below:



2. Calculate the volume of a cone with base radius R and height h, as shown below for $R = 1$ and $h = 2$



Take the base to lie in the x–y plane and begin by observing that the radius r of the cone at height z is given by

$$r(z) = \frac{R}{h}(h - z)\,.$$

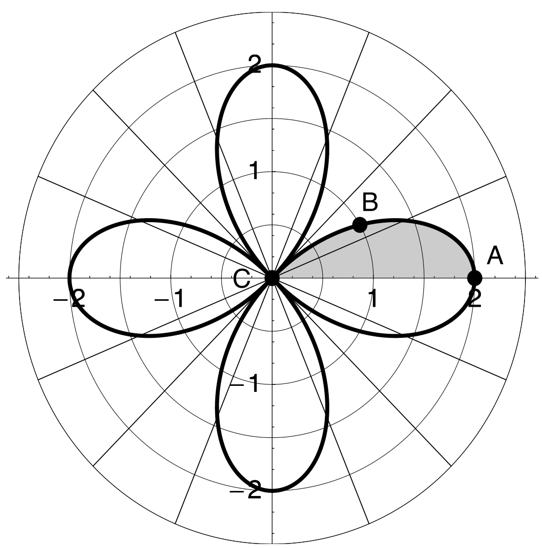
Then, construct the total volume of the cone by integrating the differential volumes $\pi r^2(z)\, dz$ at each height:

$$\iiint_V dx\, dy\, dz = \pi \int_0^h r^2(z)\, dz\,.$$

Hence, obtain

$$\iiint_V dx\, dy\, dz = \tfrac{1}{3}\pi R^2 h\,.$$

3. The figure below shows a polar plot of the function $r(\phi) = 2\cos(2\phi)$, for $0 \le \phi < 2\pi$:



The points in the figure label polar coordinates (r, ϕ) as follows:

$$\text{A}:\ (2, 0)\,, \qquad \text{B}:\ (1, \tfrac{1}{6}\pi)\,, \qquad \text{C}:\ (0, \tfrac{1}{4})\,.$$

Compute the area enclosed by this "clover leaf" in circular polar coordinates by following the procedure outlined below:

(a) The symmetry of the graph means that the total area is 8 times the area between the x-axis and the curve ABC (shown shaded in the figure). Determine the ranges of ϕ and r within this region.

(b) Show that the area A enclosed by the clover leaf is given by

$$A = 8\int_0^{\frac{1}{4}\pi} d\phi \int_0^{2\cos(2\phi)} r\, dr\,.$$

(c) Evaluate this integral to obtain

$$A = 2\pi\,.$$