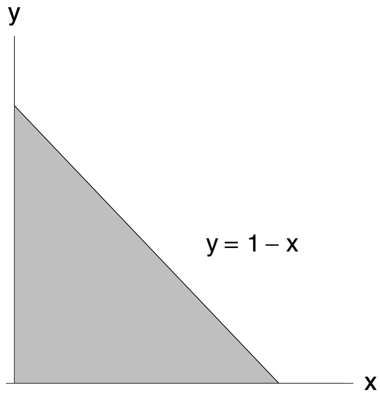
First-Year Mathematics

Solutions to Classwork 2 January 14, 2005

1. The area to be integrated is shown shaded below:



(a) For x fixed ($0 \le x \le 1$), the range of y is $0 \le y \le 1-x$ and for y fixed ($0 \le y \le 1$), the range of x is$0 \le x \le 1 - y$.

(b) In the first case in (b), the area of A is represented as

$$A = \int_0^1 dx \int_0^{1-x} dy\,,$$

where the integral is carried out first over y and then over x. Evaluating the integrals, we obtain

$$A = \int_0^1 (1-x)\,dx = x\Big|_0^1 - \tfrac{1}{2}x^2\Big|_0^1 = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

which, by inspection of the diagram in (a), is the area of A.
Alternatively, the area of A can be written as

$$A = \int_0^1 dy \int_0^{1-y} dx\,,$$

where now the integral is carried out first over x and then over y. Evaluation of this integral also yields $A = \frac{1}{2}$.

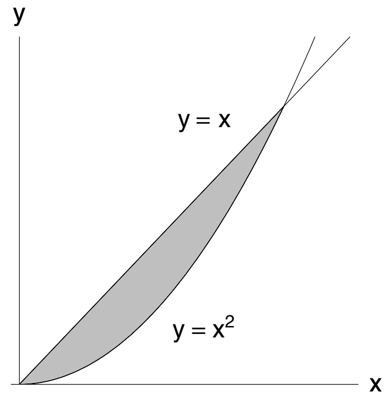
(c) Either of the integrals in (c) can be used to integrate xy over A. Thus,

$$\begin{aligned}\int_0^1 dx \int_0^{1-x} dy\, xy &= \int_0^1 x\,dx \int_0^{1-x} y\,dy \\ &= \tfrac{1}{2}\int_0^1 x(1-x)^2\,dx \\ &= \tfrac{1}{4}x^2\Big|_0^1 - \tfrac{1}{3}x^3\Big|_0^1 + \tfrac{1}{8}x^4\Big|_0^1\end{aligned}$$

$$= \tfrac{1}{4} - \tfrac{1}{3} + \tfrac{1}{8}$$

$$= \tfrac{1}{24}\,.$$

2. The area to be integrated is shown in the shaded region below:



   (a) For fixed x, the range of y is $x^2 \le y \le x$. The corresponding range of x is $0 \le x \le 1$.

   (b) Given the results in (b), the area of A is written as a double integral as

$$A = \int_0^1 dx \int_{x^2}^{x} dy\,.$$

   (c) Evaluating A, yields

$$A = \int_0^1 (x^2 - x)\,dx = \tfrac{1}{3}x^3\Big|_0^1 - \tfrac{1}{2}x^2\Big|_0^1 = \tfrac{1}{3} - \tfrac{1}{2} = \tfrac{1}{6}\,.$$

3. The integral in Part 2 can also be evaluated by reversing the order of integration. Thus, for fixed y, the range of x is $y \le x \le \sqrt{y}$ and the range of y is $0 \le y \le 1$. Thus, we can represent the area of A as a double integral in the alternative form

$$A = \int_0^1 dy \int_y^{\sqrt{y}} dx\,.$$

   Evaluating this integral yields

$$A = \int_0^1 (\sqrt{y} - y)\,dy = \tfrac{2}{3}y^{3/2}\Big|_0^1 - \tfrac{1}{2}y^2\Big|_0^1 = \tfrac{2}{3} - \tfrac{1}{2} = \tfrac{1}{6}\,.$$

   which agrees with the result in Part 2(c).

4. An alternative way of obtaining the area of the region in Parts 2 and 3 is to calculate the *difference* between the ordinary integrals of $y = x$ and $y = x^2$ between $x = 0$ and $x = 1$. Accordingly, we have

$$A = \int_0^1 x\,dx - \int_0^1 x^2\,dx = \tfrac{1}{2}x^2\Big|_0^1 - \tfrac{1}{3}x^3\Big|_0^1 = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6}\,,$$

which, as expected, is the same result obtained in Parts 2 and 3.

5. By introducing new variables x' and y' such that $x = ax'$ and $y = by'$, the equation of the ellipse becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = x'^2 + y'^2 = 1\,, \tag{1}$$

which, in the (x', y') coordinate system is the equation of a circle of unit radius centered at the origin. Thus, the area integral is transformed as follows:

$$\iint_A dx\,dy = ab\iint_{A'} dx'\,dy'\,, \tag{2}$$

where A the area of the ellipse and A' is the area of the circle. Transforming the second integral into circular polar coordinates,

$$x' = r\cos\phi\,, \qquad y' = r\sin\phi\,, \tag{3}$$

with

$$0 \le r \le 1\,, \qquad 0 \le \phi < 2\pi\,, \tag{4}$$

we have

$$\iint_{A'} dx'\,dy' = \underbrace{\int_0^1 r\,dr}_{\frac{1}{2}}\underbrace{\int_0^{2\pi} d\phi}_{2\pi} = \pi\,. \tag{5}$$

Thus, the area of an ellipse is

$$\iint_A dx\,dy = \pi ab\,. \tag{6}$$