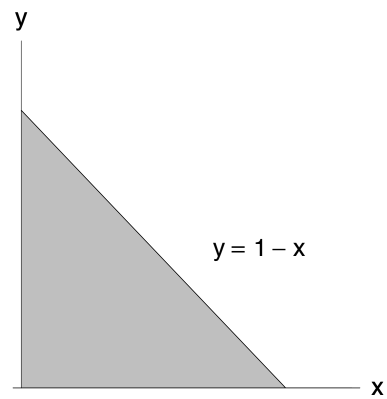
First-Year Mathematics

Classwork 2 Double Integrals January 14, 2005

1. Consider the region A in the x-y plane bounded by the x-axis, the y-axis, and the line $y = 1 - x$, shown shaded below:



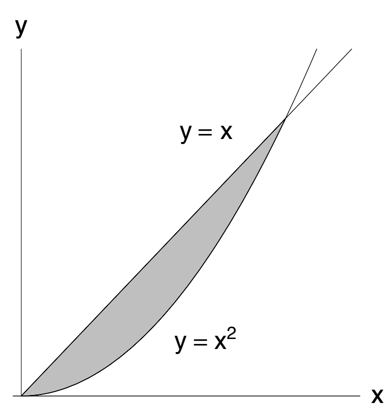
Evaluate the integral of $f(x, y) = xy$ over A by following the steps below.

(a) For a fixed value of x (or y) determine the corresponding values of y (or x).

(b) Verify that you have the correct limits by calculating the area of A.

(c) Evaluate the integral of xy over A.

2. Consider the region in the x-y plane bounded from above by $y = x$ and from below by $y = x^2$ for $0 \leq x \leq 1$, shown shaded below:



The integral

$$\iint dx\, dy$$

over this region represents its area. Compute this area by integrating first with respect to x and then with respect to y. Proceed as follows:

(a) For a fixed value of x, determine the corresponding range of y.

(b) Hence, show that the area A represented by the double integral is now given by

$$A = \int_0^1 dx \int_{x^2}^{x} dy \,.$$

(c) Evaluate this integral to obtain

$$A = \tfrac{1}{6} \,.$$

3. Evaluate the area in Part 2 by reversing the order of integration, i.e., by identifying the range of x for a fixed value of y. Show that the double integral representing the area is

$$A = \int_0^1 dy \int_y^{\sqrt{y}} dx$$

Evaluate this integral and show that the area is the same as that calculated in Part 2.

4. Obtain the area A in Parts 2 and 3 by evaluating the difference between appropriate *one-dimensional* integrals.

5. The equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \,,$$

where a and b are constants. Use circular polar coordinates to show that the area of this ellipse is πab.

Hint: First transform (x, y) into variables (x', y') where the ellipse reduces to a circle,

$$x'^2 + y'^2 = 1 \,,$$

and calculate the area of this circle taking into account the effect of the variable change on the elements of integration.