First-Year Mathematics

Solutions to Classwork 1 Derivatives and Integrals January 7, 2005

1. The derivative of x^n with respect to x is defined as

$$\frac{d\,x^n}{dx} = \lim_{\Delta x \to 0} \left[\frac{(x+\Delta x)^n - x^n}{\Delta x} \right] .$$

The expansion of $(x+\Delta x)^n$ is, according to the binomial theorem, given by

$$(x+\Delta x)^n = x^n + nx^{n-1}\Delta x + \tfrac{1}{2}n(n-1)x^{n-2}(\Delta x)^2 + \cdots .$$

Substitution of this expression into the definition of the derivative and taking the limit $\Delta x \to 0$ yields

$$\frac{d\,x^n}{dx} = \lim_{\Delta x \to 0} \left(nx^{n-1} + \tfrac{1}{2}x^{n-2}\Delta x + \cdots \right) = nx^{n-1} .$$

2. The derivative of the quantity $h(x) = af(x) + bg(x)$ is

$$\begin{aligned}
\frac{dh}{dx} &= \lim_{\Delta x \to 0} \left[\frac{h(x+\Delta x) - h(x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \left\{ \frac{1}{\Delta x} \Big[af(x+\Delta x) + bg(x+\Delta x) - af(x) - bg(x) \Big] \right\} \\
&= a \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right] + b \lim_{\Delta x \to 0} \left[\frac{g(x+\Delta x) - g(x)}{\Delta x} \right] \\
&= a \frac{df}{dx} + b \frac{dg}{dx} .
\end{aligned}$$

3. The derivative of the product of two functions fg is defined as

$$\frac{d(fg)}{dx} = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \right] .$$

By using the identity,

$$f(t)g(t) - f(x)g(x) = f(t)\big[g(t) - g(x)\big] + g(x)\big[f(t) - f(x)\big] ,$$

and making the substitution $t \to x + \Delta x$, we have

$$\begin{aligned}
\frac{d(fg)}{dx} &= \lim_{\Delta x \to 0} \left\{ f(x+\Delta x) \left[\frac{g(x+\Delta x) - g(x)}{\Delta x} \right] + g(x) \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right] \right\} \\
&= f(x)\frac{dg}{dx} + g(x)\frac{df}{dx}\,.
\end{aligned}$$

4. By setting $g = 1/f$ in the product rule for derivatives (assuming, of course, that $f(x) \neq 0$), we have that $fg = 1$. Since the derivative of a constant vanishes, we obtain

$$\frac{df}{dx}\frac{1}{f} + f\frac{d}{dx}\left(\frac{1}{f}\right) = 0\,.$$

Solving for the derivative of $1/f$ yields

$$\frac{d}{dx}\left(\frac{1}{f}\right) = -\frac{1}{f^2}\frac{df}{dx}\,.$$

5. By writing the quotient f/g as the product $f(1/g)$, we can use the product rule in Part 3 together with the result in Part 4 to write

$$\begin{aligned}
\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{d}{dx}\left(f \times \frac{1}{g}\right) &= \frac{df}{dx}\frac{1}{g} + f\frac{d}{dx}\left(\frac{1}{g}\right) \\
&= \frac{df}{dx}\frac{1}{g} - \frac{f}{g^2}\frac{dg}{dx} \\
&= \frac{1}{g^2}\left(\frac{df}{dx}g - f\frac{dg}{dx}\right)\,.
\end{aligned}$$

6. Integration by parts proceeds by writing

$$\begin{aligned}
\int_a^b \cos^2 x\,dx = \int_a^b \underbrace{\cos x}_{u}\,\underbrace{\cos x\,dx}_{dv} &= \sin x \cos x\bigg|_a^b + \int_a^b \sin^2 x\,dx \\
&= \sin x \cos x\bigg|_a^b + \int_a^b (1 - \cos^2 x)\,dx \\
&= \sin x \cos x\bigg|_a^b + \int_a^b dx - \int_a^b \cos^2 x\,dx\,.
\end{aligned}$$

Rearranging, we obtain

$$\begin{aligned}\int_a^b \cos^2 x\,dx &= \tfrac{1}{2}\int_a^b dx + \tfrac{1}{2}\sin x\cos x\Big|_a^b \\ &= \tfrac{1}{2}(b-a) + \tfrac{1}{2}(\sin b\cos b - \sin a\cos a)\,.\end{aligned}$$

Now, by using the trigonometric identity $\cos(2x) = 2\cos^2 x - 1$, the integral to be evaluated is

$$\begin{aligned}\int_a^b \cos^2 x\,dx &= \tfrac{1}{2}\int_a^b dx + \tfrac{1}{2}\int_a^b \cos(2x)\,dx \\ &= \tfrac{1}{2}x\Big|_a^b + \tfrac{1}{4}\sin(2x)\Big|_a^b \\ &= \tfrac{1}{2}(b-a) + \tfrac{1}{4}[\sin(2b) - \sin(2a)] \\ &= \tfrac{1}{2}(b-a) + \tfrac{1}{2}(\sin b\cos b - \sin a\cos a)\end{aligned}$$

Since $\sin(n\pi) = 0$, for any integer n, we have that

$$\int_0^{n\pi} \cos^2 x\,dx = \tfrac{1}{2}n\pi\,.$$