# First-Year Mathematics

Classwork 1 | Derivatives and Integrals | January 7, 2005

**1.** Use the definition of the derivative to prove the following formula:

$$\frac{d\,x^n}{dx} = nx^{n-1}\,,$$

for any positive integer $n$.

**2.** Show that differentiation is a *linear* operation, i.e. show that, for any two differentiable functions $f$ and $g$,

$$\frac{d}{dx}\big[af(x) + bg(x)\big] = a\frac{df}{dx} + b\frac{dg}{dx}\,,$$

where $a$ and $b$ are constants.

**3.** Suppose that we have two differentiable functions $f(x)$ and $g(x)$. The derivative of their *product* $fg$ is defined as

$$\frac{d(fg)}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}\right].$$

Use the identity

$$f(t)g(t) - f(x)g(x) = f(t)\big[g(t) - g(x)\big] + g(x)\big[f(t) - f(x)\big]\,,$$

together with an appropriate identification of $t$, to obtain the well-known product rule for derivatives:

$$\frac{d\,(fg)}{dx} = \frac{df}{dx}g + f\frac{dg}{dx}\,.$$

**4.** Use the result of Part 3 to deduce that

$$\frac{d}{dx}\left[\frac{1}{f(x)}\right] = -\frac{1}{f^2}\frac{df}{dx}\,.$$

Hint: Take $g = 1/f$ in the product rule for derivatives.

**5.** Use the result of Parts 3 and 4 to obtain the derivative of the *quotient* of two differentiable functions $f$ and $g$:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{1}{g^2}\left(\frac{df}{dx}g - f\frac{dg}{dx}\right).$$

**6.** Evaluate the following indefinite integral,

$$\int_a^b \cos^2 x\, dx$$

by using integration by parts to obtain

$$\int_a^b \cos^2 x\, dx = \tfrac{1}{2}(b-a) + \tfrac{1}{2}(\sin b \cos b - \sin a \cos a).$$

Show that this result can also be obtained by using the trigonometric identity

$$\cos(2x) = 2\cos^2 x - 1.$$

Determine the value of the integral

$$\int_0^{n\pi} \cos^2 x\, dx,$$

where $n$ is any positive integer.