Chapter 8

Ordinary Differential Equations

Many physical phenomena are described by a function whose value at a given point depends on its values at neighboring points. Thus, the equation determining this function contains derivatives of the function, such as a first derivative to indicate the slope or a velocity, a second derivative to indicate the curvature or an acceleration, and so on. Such an equation, which establishes a relation between the function and its derivatives, is called a **differential equation**.

Differential equations fall into one of two basic categories that are distinguished by the number of independent variables for the function in question. A differential equation for a function of a single independent variable contains only *ordinary* derivatives of that function and is called an **ordinary differential equation**. A differential equation for a function of two or more independent variables contains *partial* derivatives of the function and therefore is called a **partial differential equation**.

The fundamental equations at the heart of almost all areas of science and engineering are expressed as differential equations. Among the best known of these are Newton's second law of motion in mechanics, Maxwell's equations in electromagnetism, Schrödinger's equation and Dirac's equations in quantum mechanics, the Navier–Stokes equation in fluid mechanics and aerodynamics, Einstein's equations in general relativity, the Fokker–Planck equation in nonequilibrium statistical mechanics, the Hodgkin–Huxley equation in cellular biology, and the Black–Scholes equation in quantitative finance. The widespread use of differential equations is evident in many aspects of modern life, including weather prediction, transportation, communication, and macroeconomic forecasting, to name just a few. In all of these cases, the differential equations embody the characteristics of specific natural or social phenomena, often manifesting unexpected complexity, which are most clearly revealed by examining their solutions in particular cases.

8.1 Notation and Nomenclature

An ordinary differential equation for a function y of a single independent variable x is a functional relationship between x, y and the derivatives of y. The most general form of an ordinary differential equation can thus be written as

$$F(x, y, y', y'', \ldots) = 0\,, \tag{8.1}$$

where F is a known function and the primes in the argument list of F signify derivatives of y with respect to x:

$$y' \equiv \frac{dy}{dx}, \quad y'' \equiv \frac{d^2y}{dx^2}, \cdots\,. \tag{8.2}$$

When using primes is inconvenient, the nth derivative of y is indicated with a superscript:

$$y^{(n)} \equiv \frac{d^n y}{dx^n}\,. \tag{8.3}$$

A **solution** y of (8.1) is an expression which, when substituted into (8.1), results in an identity.

The **order** of a differential equation is the order of the highest derivative appearing in the argument list of F in (8.1). For example, the most general form of a first-order ordinary differential equation is

$$F(x, y, y') = 0\,. \tag{8.4}$$

The general form of an nth-order ordinary differential equation is therefore given by the expression

$$F\Big[x, y, y', \ldots, y^{(n)}\Big] = 0\,. \tag{8.5}$$

If the function F in these equations is a polynomial in the highest-order derivative of y appearing in its argument list, then the **degree** of the differential equation is the power to which this highest derivative is raised, i.e. the degree of that polynomial. An equation is said to be **linear** if F is of first degree in y and in each of the derivatives appearing as arguments of F. Thus, the general form of a linear nth-order ordinary differential equation is

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y(x) = f(x)\,, \tag{8.6}$$

where $f(x)$ and the coefficients $a_1(x), \ldots, a_n(x)$ are known functions. If $f = 0$ this differential equation is said to be **homogeneous**; otherwise, it is **inhomogeneous**.

Classifying differential equations according to order and degree is useful because the methods that are appropriate for solving equations depend very strongly on these quantities. The solution of linear equations, in particular, benefits from a vast analytic (and computational) methodology that began with the work of Newton and Leibniz in the 17th century. The situation for nonlinear and quasi-linear equations is substantially different. Although there is abundant work on particular equations, there is not the wealth of general analytic techniques that are available for solving linear equations. The reason for this is the superposition principle for linear equations, which means that any linear combination of solutions of a linear equation is also a solution of that equation. This facilitates immense flexibility in constructing general solutions from superpositions of elementary solutions.

8.2 First-Order Equations

8.2.1 Radioactive Decay

We begin our discussion with first-order equations. Our first example is based on the phenomenon of radioactive decay. We denote by $Q(t)$ the amount of material present at time t. This material decays at a rate r proportional to the amount of material present. The differential equation that describes this process is

$$\frac{dQ}{dt} = -rQ\,, \tag{8.7}$$

where the minus sign indicates that the amount of material decreases with time. We will solve this equation, which is a *linear* equation, by using two standard methods.

Method 1: Trial solution

We attempt to solve this equation with a solution of the form $Q(t) = e^{mt}$, where m is to be determined. Substituting this expression into Eq. (8.7), we find

$$\frac{dQ}{dt} = me^{mt} = -re^{mt}\,, \tag{8.8}$$

or,

$$(m+r)e^{mt} = 0\,. \tag{8.9}$$

Thus, we can obtain a solution if we set $m = -r$ (since the exponential is nonzero for finite x and finite m). The most general solution we can write for Eq. (8.7) is, therefore,

$$Q(t) = Ae^{-rt}\,, \tag{8.10}$$

where A is *any* constant. We can determine A by appealing to the physical situation described by our differential equation. If we set $t = 0$, then $Q(0)$ should correspond to the amount of material initially present, which we denote by Q_0. Accordingly,

$$Q(0) = A = Q_0\,. \tag{8.11}$$

Thus, the solution of Eq. (8.7) for the amount of material at time t is

$$\boxed{Q(t) = Q_0e^{-rt}\,.} \tag{8.12}$$

This shows that the solution is obtained not just by solving the differential equation, but by also using **initial conditions** that are appropriate for a particular set of circumstances.

Method 2: Separation of Variables

This method proceeds by rearranging Eq. (8.7) as

$$\frac{dQ}{Q} = -r\,dt\,. \tag{8.13}$$

Because the dependent variable (Q) appears only on the left-hand side of the equation, and the independent variable (t) appears only on the right-hand side, these variables are said to have been *separated* and the resulting equation can be integrated directly. Thus, with $Q(0) = Q_0$, we have

$$\int_{Q_0}^{Q(t)} \frac{dQ'}{Q'} = -r \int_0^t dt' \,. \tag{8.14}$$

Integrating, we obtain

$$\ln Q' \Big|_{Q_0}^{Q(t)} = \ln \left[\frac{Q(t)}{Q_0} \right] = -rt \,, \tag{8.15}$$

or, upon solving for $Q(t)$,

$$Q(t) = Q_0 e^{-rt} \,, \tag{8.16}$$

as above. This solution is plotted in Fig. 8.1. The characteristic exponential decay is clearly evident. With increasing r, the rate of decay is considerably faster because this factor appears in the argument of an exponential function.

The advantage of the trial solution method is that it can be applied to higher-order equations, as we will show in Sec. 8.3.1, but only to *linear* equations. The separation of variables method can be applied to certain types of nonlinear equations, but only to *first-order* equations.

8.2.2 Spread of Epidemics

A timely example of the use of differential equations to model the spread of epidemics, first used by Daniel Bernoulli in 1760 to model the spread of smallpox. We will construct a simple model of an epidemic and then solve the resulting differential equation.

Consider a population that is divided into two groups: a fraction x that has no disease, but is susceptible, and a fraction y that can have the disease and can infect others. We suppose that everyone belongs to one of these groups, so $x + y = 1$. We now make three assumptions about how the disease is spread:

1. The disease spreads only by direct contact between infected and uninfected individuals.

2. The fraction of infected individuals increases at a rate proportional to such contacts.

3. Both groups move freely among one another, so the number of contacts is xy.

The differential equation that embodies these assumptions is

$$\frac{dy}{dt} = \alpha xy = \alpha(1-y)y\,, \tag{8.17}$$

where α is a constant that specifies the "efficiency" of the spreading at the point of contact, and we used the fact that $x + y = 1$. We must supplement this equation with the fraction of infected individuals initially: $y(0) = y_0$.

The differential equation we have derived is a first-order *nonlinear* equation. Thus, we cannot use the trial solution method as formulated above. However, since the differential equation can be arranged as

$$\frac{dy}{y(1-y)} = \alpha dt\,, \tag{8.18}$$

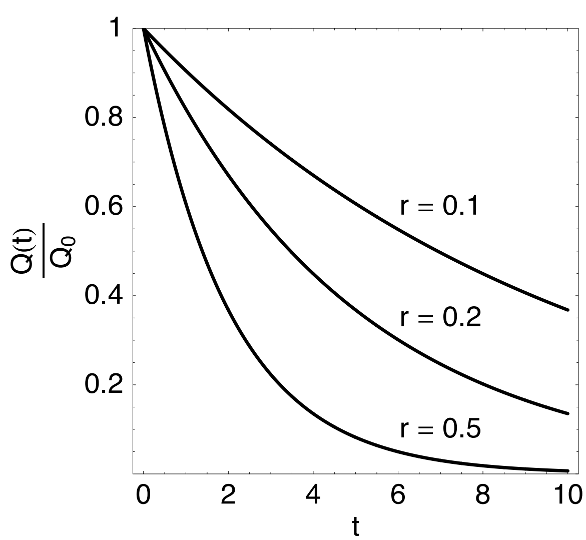

Figure 8.1: The solution in Eq. (8.16) plotted as $Q(t)/Q_0$ against t for three values of the rate constant r. With increasing r, the amount of material at time t decreases substantially.

we can use the separation of variables method. Integrating, we obtain

$$\int_{y_0}^{y(t)} \frac{dy'}{y'(1-y')} = \alpha \int_0^t dt' \,. \tag{8.19}$$

The left-hand side of this equation can be integrated by the method of partial fractions:

$$\begin{aligned} \alpha t &= \int_{y_0}^{y(t)} \frac{dy'}{y'} + \int_{y_0}^{y(t)} \frac{dy'}{1-y'} \\ &= \ln y' \Big|_{y_0}^{y(t)} - \ln(1-y') \Big|_{y_0}^{y(t)} \\ &= \ln \left[\frac{y(t)}{1-y(t)} \frac{1-y_0}{y_0} \right] . \end{aligned} \tag{8.20}$$

Solving for $y(t)$ yields,

$$\boxed{y(t) = \frac{y_0\, e^{\alpha t}}{1 - y_0(1-e^{\alpha t})}\,.} \tag{8.21}$$

As $t \to \infty$, $y(t) \to 1$, provided that $y_0 \neq 0$ (Fig. 8.2). In other words, all of the population eventually becomes infected unless there is no infection initially. As long as $y_0 \neq 0$, no matter how small, the entire population becomes infected. Accordingly, the point $y = 1$ is said to be **stable** and the point $y = 0$ is said to be **unstable**.

8.3 Equations with Constant Coefficients

Among the simplest ordinary differential equations are linear homogeneous equations with constant coefficients, i.e. those of the form (8.6) where the $a_k(x)$ $(k = 0, 1, \ldots, n)$ are constants and $f = 0$. We illustrate the solution for such equations with an application to second-order equations. The generalization to higher-order equations is straightforward.

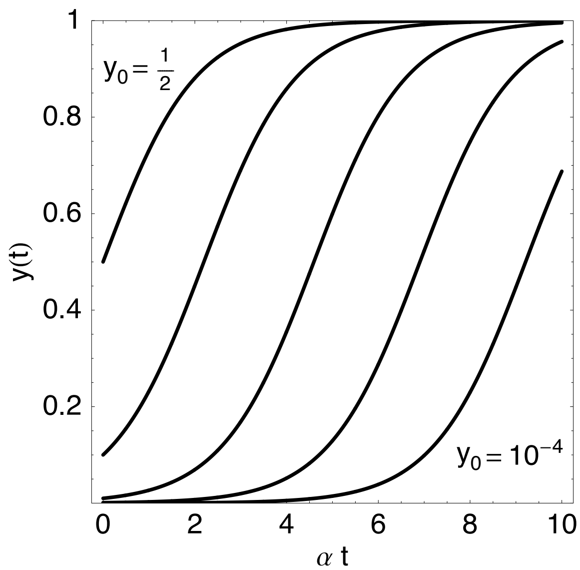

Figure 8.2: The solution in Eq. (8.21) shown as a function of αt for values of y_0 in the range $10^{-4} \le y_0 \le \frac{1}{2}$. As y_0 decreases toward zero, the solution remains near $y = 0$ for longer times, while as y_0 increases toward unity, the solution approaches $y = 1$ for shorter times.

8.3.1 The Characteristic Equation

The most general second-order linear homogeneous ordinary differential equation with constant coefficients is

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + c\,y = 0\,, \tag{8.22}$$

where a, b and c are known real constants. The recursive property of derivatives of the exponential function,

$$\frac{d^n}{dx^n}(e^{mx}) = m^n e^{mx} \tag{8.23}$$

suggests that the trial solution method used for solving first-order equations above can be applied to higher-order equations. Suppose we try this for Eq. (8.22). We substitute our trial solution e^{mx} into this equation and choose m by requiring that the resulting expression to equal zero, i.e. that this function solves the equation. Upon taking the required derivatives, we obtain

$$a\,\frac{d^2}{dx^2}(e^{mx}) + b\,\frac{d}{dx}(e^{mx}) + c\,(e^{mx}) = (am^2 + bm + c)\,e^{mx}\,. \tag{8.24}$$

For the function e^{mx} to be a solution of (8.22), the coefficient of e^{mx} on the right-hand side of this equation must vanish (since the exponential is nonzero for finite x). Thus, m must be chosen to be a root of the quadratic equation

$$\boxed{am^2 + bm + c = 0\,.} \tag{8.25}$$

This is the **characteristic equation** of the differential equation (8.22) and the left-hand side of this equation is called the **characteristic polynomial**. The roots of the characteristic equation, which are given by the quadratic formula,

$$\boxed{m = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}\,,} \tag{8.26}$$

yield solutions of (8.22). By their appearance in the discriminant in this equation, the coefficients a, b and c are seen to be the central quantities for determining the number and type of roots of the characteristic polynomial and, through these roots, the behavior of the exponential solutions. There are three cases to consider.

8.3.2 Case I: Real Distinct Roots

If $b^2-4ac > 0$, there are two distinct real roots of the characteristic equation, which we denote by m_1 and m_2. There result two distinct solutions of (8.22):

$$y_1(x) = e^{m_1 x}, \qquad y_2(x) = e^{m_2 x}\,, \tag{8.27}$$

so the most general solution of the differential equation is obtained by taken an arbitrary linear combination of these solutions:

$$\boxed{y(x) = Ae^{m_1 x} + Be^{m_2 x}\,,} \tag{8.28}$$

where A and B are arbitrary constants. This is called the **general solution** of the differential equation. The constants A and B are determined by specifying initial conditions. Depending on the signs of m_1 and m_2, these solutions exhibit either exponential growth or exponential decay.

8.3.3 Case II: Degenerate Roots

If $b^2 - 4ac = 0$, there is only a single real root, $m_1 = b/(2a)$, of the characteristic equation. Thus, this method produces only one solution of (8.22):

$$y_1(x) = e^{m_1 x}\,. \tag{8.29}$$

To obtain a second solution y_2, we return to (8.24). If we set $m = m_1$, then the right-hand side of this equation vanishes, which shows that $e^{m_1 x}$ is a solution. But suppose we differentiate both sides of this equation with respect to m *before* setting m equal to m_1. Since the order of derivatives with respect to x and to m is immaterial, we obtain

$$\begin{aligned} a\,\frac{d^2}{dx^2}(x\,e^{mx}) + b\,\frac{d}{dx}(x\,e^{mx}) + c\,(x\,e^{mx}) \\ = (2am+b)\,e^{mx} + (am^2+bm+c)x\,e^{mx}\,. \end{aligned} \tag{8.30}$$

By setting $m = m_1$, both terms on the right-hand side of this equation vanish, leaving

$$a\,\frac{d^2}{dx^2}(x\,e^{m_1 x}) + b\,\frac{d}{dx}(x\,e^{m_1 x}) + c\,(x\,e^{m_1 x}) = 0\,, \tag{8.31}$$

which shows that our second solution is, in this case,

$$y_2(x) = x\,e^{m_1 x}\,. \tag{8.32}$$

The general solution is

$$\boxed{y(x) = (A + Bx)e^{m_1 x}\,.} \tag{8.33}$$

Similar to those in (8.27), the solutions in (8.29) and (8.32) exhibit either exponential growth or decay, depending on the sign of m_1.

8.3.4 Case III: Complex Conjugate Roots

If $b^2 - 4ac < 0$, there are two complex roots, m_1 and m_2, which are complex conjugates: $m_2 = m_1^*$. The two solutions of (8.22) are thus given by

$$y_1(x) = e^{m_1 x}, \qquad y_2(x) = e^{m_1^* x}, \tag{8.34}$$

so the general solution is

$$y(x) = Ae^{m_1 x} + Be^{m_2 x}. \tag{8.35}$$

Since m_1 and m_2 are complex numbers, y_1 and y_2 are complex-valued functions. However, we can express the solutions to (8.22) solely in terms of real functions by utilizing by utilizing Theorem 1.1. With m_1 and m_2 expressed in terms of their real and imaginary parts as

$$m_1 = \alpha + i\beta, \qquad m_2 = \alpha - i\beta, \tag{8.36}$$

where α and β are real, we first write the solutions in (8.34) as

$$y_1(x) = e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos\beta x + i\sin\beta x), \tag{8.37}$$

$$y_2(x) = e^{(\alpha-i\beta)x} = e^{\alpha x}(\cos\beta x - i\sin\beta x). \tag{8.38}$$

Thus, by taking appropriate linear combinations of y_1 and y_2, we obtain two *real* solutions $\tilde{y}_1$ and $\tilde{y}_2$ of (8.22):

$$\tilde{y}_1(x) = e^{\alpha x}\cos\beta x \qquad \tilde{y}_2(x) = e^{\alpha x}\sin\beta x, \tag{8.39}$$

and the general solution becomes

$$\boxed{y(x) = Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x.} \tag{8.40}$$

These solutions show that the imaginary parts of m_1 and m_2 produce oscillatory behavior and their real parts, if nonzero, modulate this with either exponential growth or decay, as in (8.27), (8.29), and (8.32). The choice of whether to use the real solutions in (8.34) or their complex counterparts in (8.39) is largely a matter of taste and convenience.

Example. Consider the harmonic oscillator shown in Fig. 8.3, which consists of a mass m attached to a spring of stiffness k and damping γ. For any displacement from equilibrium, there are two forces acting on the spring: the gravitational force mg acting downward, and the forces of the spring $-kx$ and $-r\dot{x}$, which always act in *opposition* to the motion (which is the reason for the minus signs). Newton's second law of motion for the position x of the oscillator is thus given by

$$m\frac{d^2x}{dt^2} = -kx - r\frac{dx}{dt} - mg\,. \tag{8.41}$$

We rearrange as

$$\frac{dx^2}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x + g = 0\,, \tag{8.42}$$

where

$$\gamma = \frac{r}{m}\,, \qquad \omega_0^2 = \frac{k}{m}\,, \tag{8.43}$$

and ω_0 is the natural frequency of the oscillator. The constant factor g in this equation, which originates from the force mg due to gravity in Newton's second law, can be eliminated by shifting the position of the oscillator by $-mg/k$, which is the equilibrium position of the oscillator. We will not consider this term further, so the equation to be solved is

$$\frac{dx^2}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0\,. \tag{8.44}$$

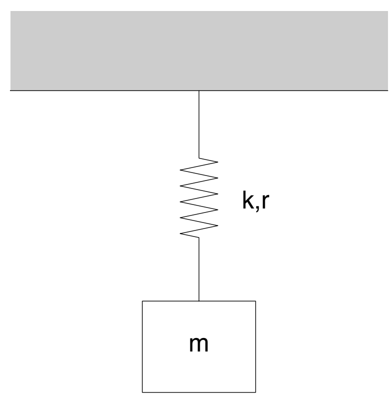

Figure 8.3: A harmonic oscillator consisting of a mass m attached to a spring of stiffness k, together with the forces acting on the mass.

To obtain a specific solution for the position of the oscillator, we must supplement this equation with two initial conditions. We take

$$x(0) = x_0\,, \qquad \left.\frac{dx}{dt}\right|_{t=0} = 0 \tag{8.45}$$

Equation (8.44) has the form of Eq. (8.22), with

$$a = 1\,, \qquad b = \gamma\,, \qquad c = \omega_0^2\,, \tag{8.46}$$

so the solutions are determined by the solving the characteristic equation (8.25), to obtain the roots

$$m = \tfrac{1}{2}\left(-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}\right) . \qquad (8.47)$$

The three cases discussed above lead to the following types of solution:

Case I. $\gamma^2 - 4\omega_0^2 > 0$. We obtain two real roots m_1 and m_2 given by

$$m_1 = \tfrac{1}{2}\left(-\gamma - \sqrt{\gamma^2 - 4\omega_0^2}\right) , \qquad (8.48)$$

$$m_2 = \tfrac{1}{2}\left(-\gamma + \sqrt{\gamma^2 - 4\omega_0^2}\right) , \qquad (8.49)$$

and the general solution is

$$x(t) = Ae^{m_1 t} + Be^{m_2 t} . \qquad (8.50)$$

The initial conditions in Eq. (8.45),

$$x(0) = A + B = x_0 , \qquad x'(0) = m_1 A + m_2 B = 0 , \qquad (8.51)$$

yield

$$A = \frac{-m_2 x_0}{m_1 - m_2} , \qquad B = \frac{m_1 x_0}{m_1 - m_2} . \qquad (8.52)$$

The solution for the position of the oscillator is therefore obtained as

$$\boxed{x(t) = \frac{x_0}{m_1 - m_2}\left(m_1 e^{m_2 t} - m_2 e^{m_1 t}\right) .} \qquad (8.53)$$

Because this solution is dominated by the damping term γ in the discriminant, it is called **over-damped** (Fig. 8.4).

Case II. $\gamma^2 - 4\omega_0^2 = 0$. We obtain a single real root m_1,

$$m_1 = -\tfrac{1}{2}\gamma , \qquad (8.54)$$

and the general solution is

$$x(t) = (A + Bt)e^{-\frac{1}{2}\gamma t} . \tag{8.55}$$

The initial conditions in Eq. (8.45),

$$x(0) = A = x_0 , \qquad x'(0) = B - \tfrac{1}{2}\gamma = 0 , \tag{8.56}$$

yield

$$A = x_0 , \qquad B = \tfrac{1}{2}x_0\gamma . \tag{8.57}$$

The solution for the position of the oscillator is therefore obtained as

$$\boxed{x(t) = x_0 \left(1 + \tfrac{1}{2}\gamma t\right) e^{-\frac{1}{2}\gamma t} .} \tag{8.58}$$

Because this solution is obtained by the balance of the damping term γ with the oscillating term ω_0 in the discriminant, it is called **critically-damped** (Fig. 8.4).

Case III. $\gamma^2 - 4\omega_0^2 < 0$. We obtain two roots m_1 and m_2 that are complex conjugates, given by

$$m_1 = \tfrac{1}{2}\left(-\gamma - i\sqrt{4\omega_0^2 - \gamma^2}\right) , \tag{8.59}$$

$$m_2 = \tfrac{1}{2}\left(-\gamma + i\sqrt{4\omega_0^2 - \gamma^2}\right) , \tag{8.60}$$

and the general solution is

$$x(t) = Ae^{m_1 t} + Be^{m_2 t} . \tag{8.61}$$

The initial conditions in Eq. (8.45),

$$x(0) = A + B = x_0 , \qquad x'(0) = m_1 A + m_2 B = 0 , \tag{8.62}$$

yield

$$A = \frac{-m_2 x_0}{m_1 - m_2} , \qquad B = \frac{m_1 x_0}{m_1 - m_2} . \tag{8.63}$$

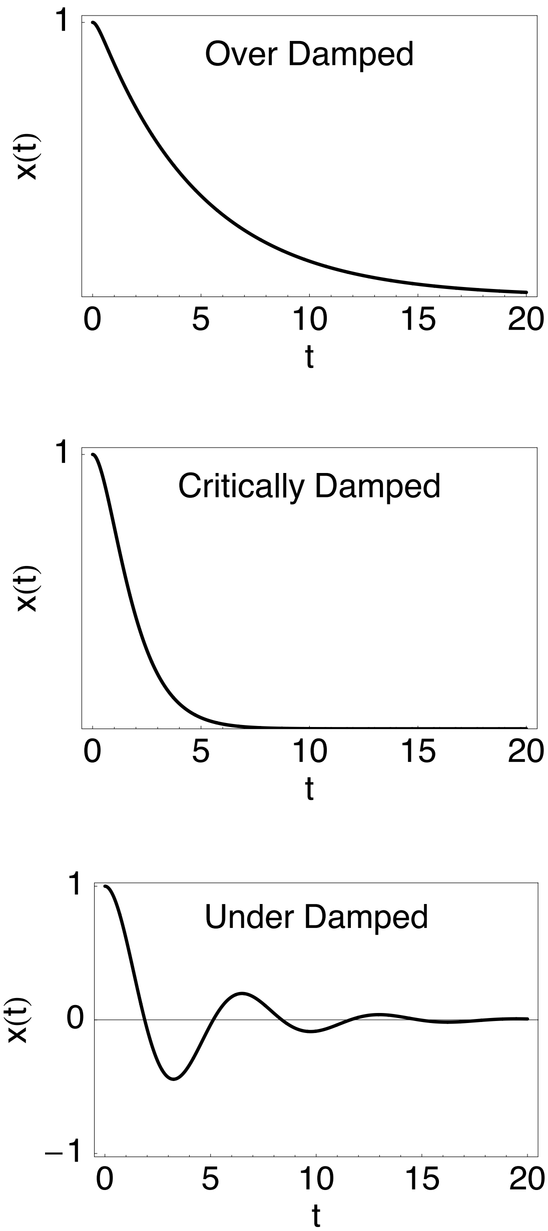

Figure 8.4: The three types of solution for a damped harmonic oscillator: over damped, critically damped, and under damped.

The solution for the position of the oscillator is therefore obtained as

$$\boxed{x(t) = \frac{x_0}{m_1 - m_2}\left(m_1 e^{m_2 t} - m_2 e^{m_1 t}\right) .} \tag{8.64}$$

Because this solution is dominated by the oscillating term ω_0 in the discriminant, it is called **under-damped** (Fig. 8.4).

8.4 Inhomogeneous Equations*

8.4.1 Method of Solution

One of the most striking manifestations of a driven system is resonance in a system subjected to a driving force. This motivates the discussion of equations of the form

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + c\,y = f(t)\,, \tag{8.65}$$

which are called **inhomogeneous** because the function $f(t)$ on the right-hand side of this equation is specified independently of the solution. Such equations are solved by first supposing that there are two independent solutions $y^{(1)}(x)$ and $y^{(2)}(x)$ of this equation:

$$a\,\frac{d^2y^{(1)}}{dx^2} + b\,\frac{dy^{(1)}}{dx} + c\,y^{(1)} = f(t)\,, \tag{8.66}$$

$$a\,\frac{d^2y^{(2)}}{dx^2} + b\,\frac{dy^{(2)}}{dx} + c\,y^{(2)} = f(t)\,. \tag{8.67}$$

If we subtract one equation from the other, say Eq. (8.66) from (8.67), we obtain

$$a\frac{d^2[y^{(2)} - y^{(1)}]}{dt^2} + b\frac{d[y^{(2)} - y^{(1)}]}{dt} + [y^{(2)} - y^{(1)}] = 0\,, \tag{8.68}$$

i.e. the difference $y^{(2)} - y^{(1)}$ is a solution of the *homogeneous* equation! Denote the general solution of the homogeneous equation by $Ay_1(x) + By_2(x)$, we conclude that

$$y^{(2)}(x) = Ay_1(x) + By_2(x) + y^{(1)}(x)\,. \tag{8.69}$$

This suggests the following method of solution. Find a solution $y_p(x)$ of the inhomogeneous equation, called a **particular solution**, by any means. The general solution $y(x)$ of the inhomogeneous equation is then given by

$$\boxed{y(x) = Ay_1(x) + By_2(x) + y_p(x)\,.} \tag{8.70}$$

8.4.2 Resonance in a Driven Harmonic Oscillator

To illustrate the solution of inhomogeneous equations, we consider an undamped harmonic oscillator driven by an external sinusoidal force:

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos\omega t\,, \tag{8.71}$$

where m is the mass, k is the spring constant, x is the position of the mass, t is the time, F_0 is the amplitude of the driving force with frequency ω. Upon dividing through by m, we can write this equation as

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)\,, \tag{8.72}$$

where $\omega_0 = (k/m)^{1/2}$ is the natural frequency of the oscillator. From the discussion in the preceding equation, we know that the solution of the corresponding *homogeneous* equation (i.e. the equation obtained by setting $F_0 = 0$), is

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t\,, \tag{8.73}$$

where A and B are arbitrary constants obtained by specifying two initial conditions (the initial position and velocity of the mass). The most general solution of the inhomogeneous equation is the sum of the general solution of the homogeneous and a particular solution $x_p(t)$ of the inhomogeneous equation:

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t + x_p(t)\,. \tag{8.74}$$

To determine $x_p(t)$ for this equation, we attempt a solution of the form

$$x_p(t) = C\cos\omega t\,. \tag{8.75}$$

The required derivatives are

$$\frac{dx_p}{dt} = -C\omega \sin \omega t, \qquad \frac{d^2 x_p}{dt^2} = -C\omega^2 \cos \omega t\,. \tag{8.76}$$

Substitution into Eq. (8.72),

$$-C\omega^2 \cos \omega t + C\omega_0^2 \cos \omega t = \frac{F_0}{m} \cos \omega t\,, \tag{8.77}$$

cancelling the common factor of $\cos(\omega t)$, and solving for C, yields

$$x_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t\,. \tag{8.78}$$

Note that, as $\omega \to \omega_0$, the solution becomes unbounded. This is called **resonance**. In the presence of damping, the solutions remain finite, but still become large when the resonance condition is fulfilled. The damping of oscillations close to resonance is an important engineering problem, as evidenced by the famous collapse of the Tacoma Narrows Bridge and, more recently by the re-design of the Millennium Bridge to incorporate damping.

The general solution to the inhomogeneous equation is therefore given by

$$\boxed{x(t) = A \cos \omega_0 t + B \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t\,.} \tag{8.79}$$

To solve the initial-value problem, we consider the initial condition corresponding to the mass being initially at rest:

$$x(0) = 0, \qquad \left.\frac{dx}{dt}\right|_{t=0} = 0\,. \tag{8.80}$$

Substituting these conditions into the general solution yields

$$x(0) = A + \frac{F_0}{m(\omega_0^2 - \omega^2)} = 0\,, \tag{8.81}$$

so,

$$A = -\frac{F_0}{m(\omega_0^2 - \omega^2)}\,, \tag{8.82}$$

and

$$\left.\frac{dx}{dt}\right|_{t=0} = \omega_0 B = 0\,, \tag{8.83}$$

which yields $B = 0$. Thus, the solution to the initial-value problem is

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\left(\cos\omega t - \cos\omega_0 t\right)\,. \tag{8.84}$$

This expression can be written in a physically more transparent form by using the trigonometric identity,

$$\cos(A - B) - \cos(A + B) = 2\sin A\sin B\,. \tag{8.85}$$

By setting $A - B = \omega t$ and $A + B = \omega_0 t$ and solving for A and B, we obtain

$$\boxed{x(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin\left[\tfrac{1}{2}(\omega_0 + \omega)t\right]\sin\left[\tfrac{1}{2}(\omega_0 - \omega)t\right]\,,} \tag{8.86}$$

which represents the solution as a frequency-dependent amplitude and two sinusoidal factors.

In the Fig. 8.5, we plot the quantity $X(t) = mx(t)/2F_0$ for $\omega_0 = 1$ and $\omega = 0.9$. The solution is oscillatory, as expected, but the most striking feature of this plot is the phenomenon of “beats”, resulting from the superposition of a high-frequency oscillation, $\sin[\frac{1}{2}(\omega_0 + \omega)t]$, and a lower frequency envelope, $\sin[\frac{1}{2}(\omega_0 - \omega)t]$.

8.5 Summary

We can both summarize and generalize the main results of this chapter as follows. The solution of *any* nth-order ordinary differential Equation (8.5),

$$F\Big[x, y, y', \ldots, y^{(n)}\Big] = 0$$

depends, in general, on n arbitrary constants $c_1, c_2, \ldots, c_n$:

$$y = \varphi(x; c_1, c_2, \ldots, c_n) \tag{8.87}$$

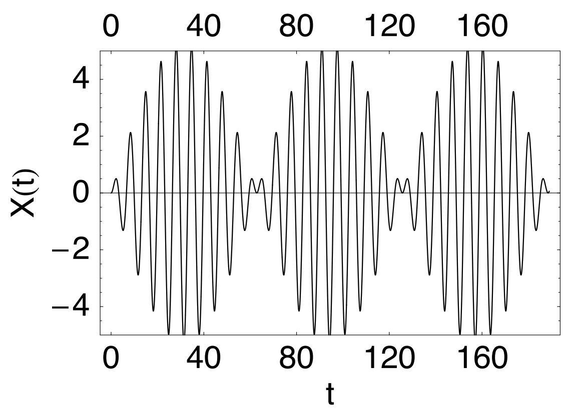

Figure 8.5: The quantity $X(t) = mx(t)/2F_0$, where $x(t)$ is the solution in Eq. (8.86), for a undamped harmonic oscillator with a natural frequency $\omega_0 = 1$ driven by a sinusoidal force with a frequency $\omega = 0.9$.

Thus, to obtain a unique solution for a particular problem, it is necessary to supplement the differential equation with auxiliary conditions. A common choice is for these constants to be determined from the initial values of the solution y and its first $n-1$ derivatives at some initial point x_0:

$$y(x_0) = A_0, \quad y'(x_0) = A_1, \quad \dots \quad y^{(n)}(x_0) = A_{n-1}$$

The expression in (8.87) is a general solution if it possible to satisfy these initial conditions for arbitrary values of the y_i with an appropriate choice of the c_j. This usually requires the solution of a system of algebraic equations.

For homogeneous linear nth-order equations,

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y(x) = 0$$

the general solution can be formed from any n linearly independent solutions $y_1, y_2, \dots, y_n$ of this equation:

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

The determination of the c_j from the initial conditions now reduces to the solution of a system of n linear algebraic equations.