Chapter 7

The Curl and Stokes' Theorem

The divergence of a vector field is a derivative operation that yields the flux density of that field as a function of position. Another quantity used to characterize a vector field is called the **curl** or, sometimes the **rotation**. The curl is obtained by taking a derivative, and is associated with the circulation density of the vector field, so there is a Fundamental Theorem of Calculus for the curl , called Stokes' theorem. In this chapter we develop these concepts by following similar steps to those used in the preceding chapter for the divergence: the definition in terms of infinitesimal quantities is followed by integration of this quantity over a region to obtain the fundamental theorem.

The divergence and curl provide complementary information about a vector field. In fact, a theorem due to Helmholtz states that a vector field that vanishes at infinity is *uniquely* defined by its curl and divergence. Maxwell's equations, the fundamental equations of electromagnetism, utilize this theorem by specifying the divergence and curl of the electric and magnetic fields.

7.1 The Curl in Two Dimensions

The basic construction used to derive the curl in two dimensions is shown in Fig. 7.1 for a vector field $\boldsymbol{V}$ given by

$$\boldsymbol{V} = P(x,y)\,\boldsymbol{i} + Q(x,y)\,\boldsymbol{j}\,. \tag{7.1}$$

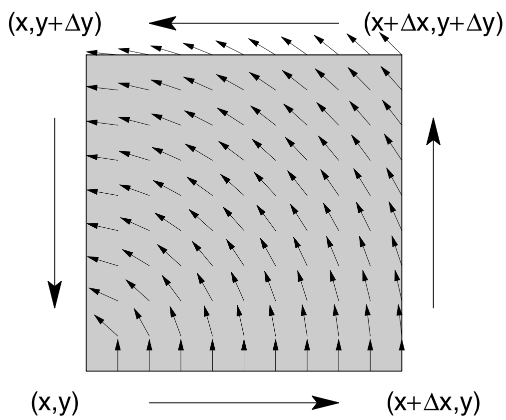

Figure 7.1: The region of area $\Delta x\Delta y$ used to calculated the curl of a vector field. The projections of the components of the vector field onto the directions around this region are indicated by arrows.

We consider a rectangular region of area $\Delta x\Delta y$ and calculate the line integral around its perimeter, which is denoted by $\partial(\Delta A)$,

$$\oint_{\partial(\Delta A)} \boldsymbol{V}\cdot d\boldsymbol{r} = \oint_{\partial(\Delta A)} (P\,dx + Q\,dy) \tag{7.2}$$

in the counterclockwise direction. The integrand of this line integral represents the projection of $\boldsymbol{V}$ along the integration path, so a positive (resp., negative) value of the integral implies that $\boldsymbol{V}$ a positive (resp., negative) circulation. The direction of positive circulation is simply a matter of convention. If the line integral vanishes, $\boldsymbol{V}$ has no circulation in the region.

The line integral in Eq. (7.2) is composed of four segments:

$$\begin{aligned}\oint_{\partial(\Delta A)} (P\,dx\,+\,Q\,dy) &= \int_x^{x+\Delta x} P(x',y)\,dx' + \int_y^{y+\Delta y} Q(x+\Delta x,y')\,dy' \\ &\quad + \int_{x+\Delta x}^{x} P(x',y+\Delta y)\,dx' + \int_{y+\Delta y}^{y} Q(x,y')\,dy' \end{aligned}\tag{7.3}$$

By using Eq. (1.30),

$$\int_a^b f(x)\,dx = -\int_b^a f(x))\,dx\,, \tag{7.4}$$

we can combine terms on the right-hand side of Eq. (7.3) to obtain

$$\oint_{\partial(\Delta A)}(P\,dx + Q\,dy) = \int_x^{x+\Delta x}\left[P(x',y) - P(x',y+\Delta y)\right]\,dx' + \int_y^{y+\Delta y}\left[Q(x+\Delta x,y') - Q(x,y')\right]\,dy'\,. \tag{7.5}$$

Since the limits $\Delta x \to$ and $\Delta y \to 0$ are to be taken, we can regard P and Q as constant over their intervals of integration. Our line integral then becomes

$$\oint_{\partial(\Delta A)}(P\,dx + Q\,dy) = \left[P(x,y) - P(x,y+\Delta y)\right]\Delta x + \left[Q(x+\Delta x,y) - Q(x,y)\right]\Delta y\,. \tag{7.6}$$

We now divide both sides of this equation by $\Delta x \Delta y$,

$$\frac{1}{\Delta x\Delta y}\oint_{\partial(\Delta A)}(P\,dx + Q\,dy) = \frac{P(x,y) - P(x,y+\Delta y)}{\Delta y} + \frac{Q(x+\Delta x,y) - Q(x,y)}{\Delta x}\,, \tag{7.7}$$

and take the limits $\Delta x \to$ and $\Delta y \to 0$. The right-hand side can be evaluated by using the definitions in Eqs. (1.12) and (1.13):

$$\lim_{\Delta y\to 0}\left[\frac{P(x,y) - P(x,y+\Delta y)}{\Delta y}\right] = -\frac{\partial P}{\partial y}\,, \tag{7.8}$$

$$\lim_{\Delta x\to 0}\left[\frac{Q(x+\Delta x,y) - Q(x,y)}{\Delta x}\right] = \frac{\partial Q}{\partial x}\,. \tag{7.9}$$

Thus, we obtain

$$\boxed{\lim_{\Delta x\to 0,\Delta y\to 0}\left[\frac{1}{\Delta x\Delta y}\oint_{\partial(\Delta A)}(P\,dx + Q\,dy)\right] = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,.} \tag{7.10}$$

This is the definition of the curl. It represents the circulation of a vector field around an infinitesimal area at (x, y). Notice that in deriving this quantity, we have used only the components *along* $d\boldsymbol{r}$, whereas the corresponding derivation of the divergence in Sec. 6.1 used the *normal* components. This provides a intuitive basis for understanding why the divergence and curl uniquely specify a vector field.

Example. We calculate the curls of the vector fields in the examples in Sec. 6.1. Consider

$$\boldsymbol{V} = x\,\boldsymbol{i} + y\,\boldsymbol{j}\,, \qquad (7.11)$$

which is shown in Fig. 7.2. This is a radial vector field with a divergence that was calculated as $\boldsymbol{\nabla}\cdot\boldsymbol{V} = 2$. To calculate the curl of this vector field, we apply the definition in Eq. (7.10) with $P = x$ and $Q = y$ to obtain

$$\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} = 0 - 0 = 0\,. \qquad (7.12)$$

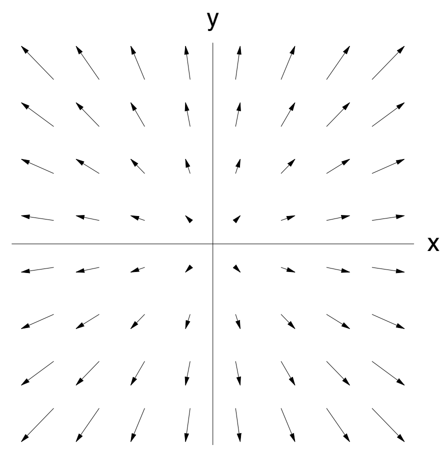

Figure 7.2: Plot of $\boldsymbol{V} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$.

Thus, $\boldsymbol{V}$ is specified completely by its divergence. These conclusions are valid for *any* radial vector field (Problem Set 9).

Now consider the vector field

$$\boldsymbol{V} = x\,\boldsymbol{i} - y\,\boldsymbol{j}\,, \qquad (7.13)$$

which is shown in Fig. 7.3. This vector field was found in Sec. 6.1 to have a divergence that vanishes: $\boldsymbol{\nabla}\cdot\boldsymbol{V} = 0$. With $P = x$ and $Q = -y$, the curl of $\boldsymbol{V}$ is

$$\frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} = 0 + 0 = 0\,, \qquad (7.14)$$

which is also zero! Thus, *both* the divergence and curl vanish for this vector field.

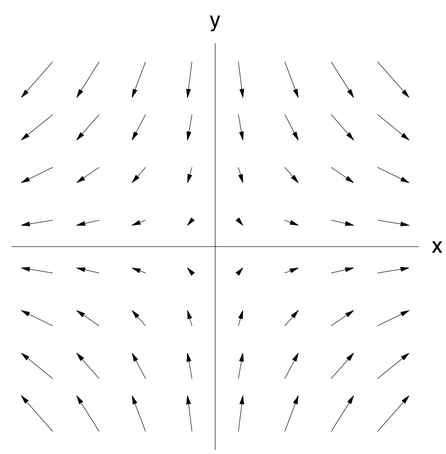

Figure 7.3: Plot of $\boldsymbol{V} = x\,\boldsymbol{i} - y\,\boldsymbol{j}$.

This example shows that, even if the curl and divergence of a vector field are both zero, the vector field itself need not reduce to a constant everywhere. However, if we further stipulate further that a vector field must vanish at infinity, then a vanishing divergence and curl do indeed imply that that $\boldsymbol{V} = 0$. This discussion has important implications for the governing equations of electric and magnetic fields in electromagnetism (Maxwell's equations).

Finally, consider

$$\boldsymbol{V} = -y\,\boldsymbol{i} + x\,\boldsymbol{j}\,, \tag{7.15}$$

which is shown in Fig. 6.4. With $P = -y$ and $Q = x$, the curl is calculated as

$$\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} = 1 + 1 = 2\,. \tag{7.16}$$

so this vector field has a positive circulation according to our convention. Indeed, the plot of the vector field in Fig. 7.4 is suggestive of a circulation around the origin. But, as in the discussion in Sec. 6.1, the fact that the curls of the vector fields in this example are constants, means that any interpretation assigned to them must be valid for every point in the x-y plane. This point is taken up in Problem Set 9. ∎

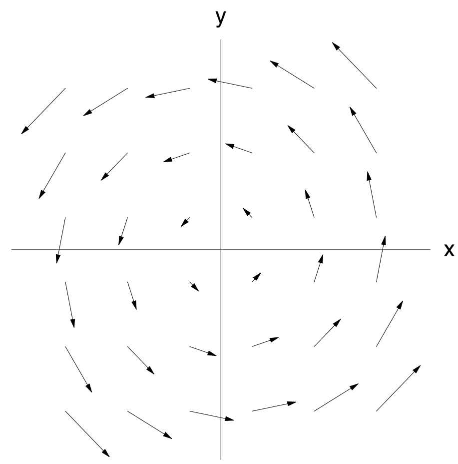

Figure 7.4: Plot of $\boldsymbol{V} = -y\,\boldsymbol{i} + x\,\boldsymbol{j}$.

7.2 Green's Theorem

The curl was obtained by considering the circulation around an infinitesimal region in the x-y plane. We can integrate Eq. (7.10) to obtain the "Fundamental Theorem of Calculus" associated with the curl in a manner analogous to that for the divergence. The key point again is that for adjacent elemental regions the share a boundary, the integral along one boundary exactly cancels the corresponding integral around the adjacent region because the sense of integration is opposite on these faces (Figs. 7.5). Thus, the only *net* contribution to the curl is where there is no adjacent region.

We begin by rearranging (7.10) as

$$\oint_{\partial(\Delta A)} (P\,dx + Q\,dy) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\Delta A\,. \tag{7.17}$$

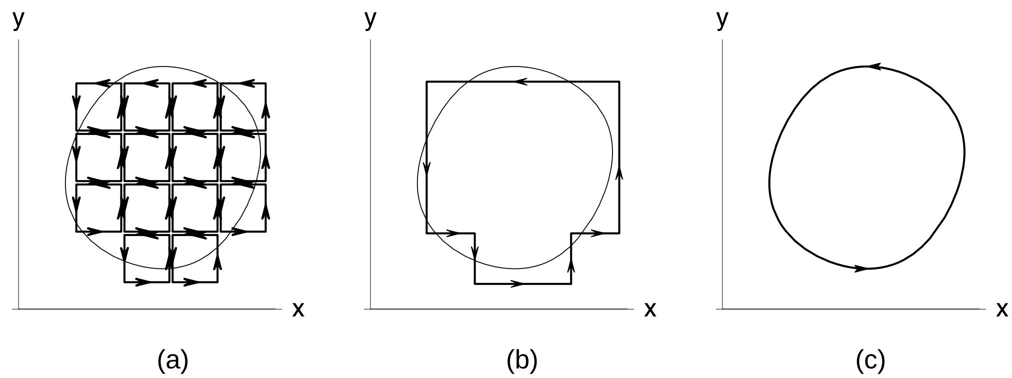

Figure 7.5: (a) Partition of a region in the x-y plane bounded by a curve. (b) Cancellation of line integrals over adjacent regions. (c) As the area of the basic regions becomes smaller ($\Delta x \to 0$ and $\Delta y \to 0$), the partitioning in (a) provides a successively more accurate representation of the region, yielding a more accurate representation of the curve surrounding the region.

The left-hand side of this equation is the line integral over the perimeter of the region ΔA and the right-hand side is the circulation density (i.e. the curl) multiplied by the area ΔA of the region, which yields the total circulation within the region. Now, any region in the x-y plane can be partitioned into such contiguous elemental rectangular regions, as shown in Fig. 7.5(a). For each such region we can carry out the indicated calculations in Eq. (7.17). Thus, the corresponding quantities calculated for the entire region A is obtained by summing over the elemental regions:

$$\sum_i \oint_{\partial(\Delta A_i)} (P\, dx + Q\, dy) = \sum_i \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \Delta A_i \,, \tag{7.18}$$

where ΔA_i is the area of the ith elemental region. This is depicted in Fig. 7.5(b). For extended regions ΔA, the partitioning does not give an accurate representation of either interior or the boundary, but improves as ΔA decreases. Thus, by taking $\Delta A \to 0$, we can obtain an exact relation between the line integral over the *perimeter* of the region and the curl *within* the region.

We consider the left-hand side of Eq. (7.18) first. As Figs. 7.5(b,c) show, the sum of the line integrals over the elemental rectangular regions contains only those segments that have no neighboring elemental regions; the interior line integrals cancel pairwise. Thus,

$$\lim_{\Delta A_i \to 0} \left[\sum_i \oint_{\partial(\Delta A_i)} (P\,dx + Q\,dy) \right] = \oint_{\partial A} (P\,dx + Q\,dy)\,, \tag{7.19}$$

where ∂A is the perimeter of the region A in the x-y plane and the integral is taken in the counter-clockwise direction.

The right-hand side of Eq. (7.18) is two-dimensional Riemann sum (cf. Eq. 1.16). In particular, as $\Delta \to 0$, the summation becomes an integral over the interior of the region A:

$$\lim_{\Delta A_i \to 0} \left[\sum_i \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \Delta A_i \right] = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA\,. \tag{7.20}$$

Thus, the results in Eqs. (7.19) and (7.20) show that the limiting form of Eq. (7.18) obtained as $\Delta A_i \to 0$ is

$$\boxed{\oint_{\partial A} (P\,dx + Q\,dy) = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy\,,} \tag{7.21}$$

which is known as **Green's theorem**. This equation has the structure of the Fundamental Theorem of Calculus in that the integral of a derivative of a quantity (the curl) over the *interior* of a region is equal to that quantity evaluated on the *boundary* of the region.

Example. Consider the vector field

$$\boldsymbol{V} = -y\,\boldsymbol{i} + x\,\boldsymbol{j}\,, \tag{7.22}$$

which is shown in Fig. 6.4, and suppose that the area in the x-y plane is a circle of radius R. With $P = -y$ and $Q = x$, the left-hand side of Eq. (7.21) is

$$\oint_{\partial A} (P\,dx + Q\,dy) = \oint_{\partial A} (x\,dy - y\,dx)\,. \tag{7.23}$$

In circular polar coordinates, $x = R\cos\phi$, $y = R\sin\phi$, where $0 \le \phi < 2\pi$, and

$$dx = -R\sin\phi\,d\phi\,, \qquad dy = R\cos\phi\,d\phi\,, \tag{7.24}$$

so the integral becomes

$$\oint_{\partial A}(x\,dy - y\,dx) = \int_0^{2\pi}(R^2\cos^2\phi + R^2\sin^2)\,d\phi$$
$$= R^2\int_0^{2\pi} d\phi = 2\pi R^2\,. \tag{7.25}$$

To evaluate the right-hand side of Eq. (7.21), we have that the curl is

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2\,, \tag{7.26}$$

so, again using polar coordinates,

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = 2\int_0^R r\,dr\int_0^{2\pi} d\phi$$
$$= 2 \times \frac{1}{2}R^2 \times 2\pi = 2\pi R^2\,, \tag{7.27}$$

which agrees with Eq. (7.25). ▮

Example. Consider the vector field

$$\boldsymbol{V} = x\,\boldsymbol{i} + y\,\boldsymbol{j}\,, \tag{7.28}$$

which is shown in Fig. 7.2. With $P = x$ and $Q = y$, the left-hand side of Eq. (7.21) is

$$\oint_{\partial A}(P\,dx + Q\,dy) = \oint_{\partial A}(x\,dx + y\,dy)\,. \tag{7.29}$$

Since the curl of $\boldsymbol{V}$ vanishes,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - 0 = 0\,. \tag{7.30}$$

Thus, according to Green's theorem,

$$\oint_{\partial A}(x\,dx + y\,dy) = 0 \tag{7.31}$$

for any area A! This ostensibly surprising result is, in fact, to be expected from the discussion in Sec. 4.2, where we showed that the value of a line integral is independent of the path if and only if the integral over any closed curve vanishes. Indeed, we showed in Sec. 4.3 that the condition for the value of a line integral,

$$\oint_{\partial A} (P\,dx + Q\,dy) \tag{7.32}$$

to be independent of the path is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\,, \tag{7.33}$$

i.e. the vanishing curl of the vector field $\boldsymbol{V} = P\,\boldsymbol{i} + Q\,\boldsymbol{j}$. The three-dimensional generalization of this result will be derived in the next section. ▮

7.3 Stokes' Theorem

Green's theorem establishes a relationship between the curl of a two-dimensional vector field and line integrals of that vector field in the x-y plane. The motivation for the mathematical structure of line integrals in terms of the work done by a force field along a path (Sec. 4.1) suggests that line integrals in three dimensions might also benefit from the analysis in Secs. 7.1 and 7.2. In this section, we generalize Green's theorem to three dimensions by examining at each side of Eq. (7.21) separately.

7.3.1 Line Integrals in Three Dimensions

Consider a three-dimensional vector field $\boldsymbol{V}$ given by

$$\boldsymbol{V} = P(x,y,z)\,\boldsymbol{i} + Q(x,y,z)\,\boldsymbol{j} + R(x,y,z)\,\boldsymbol{k}\,. \tag{7.34}$$

The work done along a three-dimensional path $\mathcal{P}$ is obtained by calculating the component of $\boldsymbol{V}$ projected along the direction of the path. With the position vector given by $\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$, we have

$$\int_{\mathcal{P}} \boldsymbol{V} \cdot d\boldsymbol{r} = \int_{\mathcal{P}} (P\,dx + Q\,dy + R\,dz)\,. \tag{7.35}$$

This is the generalization of the left-hand side of Eq. (7.21) to three dimensions.

7.3.2 The Curl Vector

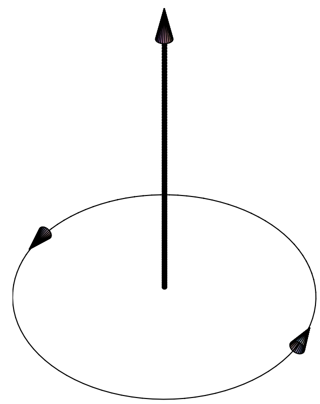

Figure 7.6: The curl vector according to the right-hand rule.

The curl derived in Sec. 7.1 is endowed with a sign in that the counterclockwise direction of the circulation is taken as positive by convention. But when this construction is extended to three dimensions, the concept of "clockwise" versus "counterclockwise" is not precise enough to identify the direction of circulation. For example, the counterclockwise direction observed by looking down onto the x-y plane from the positive z-axis appears as the *clockwise* direction when looking up to the x-y plane from the negative z-axis. This ambiguity can be alleviated by using the "right-hand rule" to assign an orientation to positive (i.e. counterclockwise) circulation: when the fingers of your right hand bend in the counterclockwise direction, your thumb points in the direction of the positive z-axis. Thus, we can write the curl of a vector $\boldsymbol{V} = P\,\boldsymbol{i} + Q\,\boldsymbol{j}$ as

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}\,. \tag{7.36}$$

We can represent this expression in a more suggestive form by utilizing the definition of the "del" operation,

$$\boldsymbol{\nabla} = \boldsymbol{i}\,\frac{\partial}{\partial x} + \boldsymbol{j}\,\frac{\partial}{\partial y} + \boldsymbol{k}\,\frac{\partial}{\partial z}\,, \tag{7.37}$$

to represent the curl of a vector as the "cross" product between this operation and $\boldsymbol{V}$. The calculation of this quantity proceeds in direct analogy with the representation of the cross product of two ordinary vectors as a determinant:

$$\boldsymbol{\nabla}\times\boldsymbol{V} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & 0 \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}\,, \tag{7.38}$$

which is the same as Eq. (7.53). To convert this to a scalar quantity, we take the "dot" product of this quantity with $\boldsymbol{k}$:

$$(\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{k}=\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,. \tag{7.39}$$

Thus, combining Eqs. (7.35) and (7.39) Green's theorem can be written as

$$\boxed{\oint_{\partial A}\boldsymbol{V}\cdot d\boldsymbol{r}=\iint_A(\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{k}\,dx\,dy\,.} \tag{7.40}$$

7.3.3 The Curl of Three-Dimensional Vector Fields

The general vector field $\boldsymbol{V}$ is given by Eq. (7.34). The construction in Eq. (7.39) yield the following expression for the curl:

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{V}&=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\ \partial_x&\partial_y&\partial_z\\ P&Q&R\end{vmatrix}\\ &=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\boldsymbol{k}\,.\end{aligned} \tag{7.41}$$

The meaning of this vector field follows from that for the two-dimensional curl: it represents the circulation density, with a direction given by the right-hand rule.

Example. The curl of the vector field

$$\boldsymbol{V}=z\,\boldsymbol{i}+x\,\boldsymbol{j}+y\,\boldsymbol{k}\,, \tag{7.42}$$

is

$$\boldsymbol{\nabla}\times\boldsymbol{V}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\ \partial_x&\partial_y&\partial_z\\ z&x&y\end{vmatrix}=\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}\,. \tag{7.43}$$

The vector components of the curl of $\boldsymbol{V}$ each have unit circulation in directions given by the right-hand rule, so the total circulation is along the direction $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$. ∎

The right-hand side of Eq. (7.21) can now be expressed in terms of the local unit normal $\boldsymbol{n}$ to the surface σ as

$$\int_A (\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{k}\,dx\,dy \longrightarrow \iint (\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{n}\,d\sigma\,. \tag{7.44}$$

We thus arrive at **Stokes' theorem**:

$$\boxed{\oint_{\partial\sigma}\boldsymbol{V}\cdot d\boldsymbol{r} = \iint_\sigma (\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{n}\,d\sigma\,,} \tag{7.45}$$

where $\partial\sigma$ is the bounding curve of the surface σ. An important consequence of the structure of this equation is that, given a vector field $\boldsymbol{V}$, the left-hand side is determined completely by the bounding curve, independent of the surface σ. To appreciate the significance of this, consider the three surfaces shown in Fig. 7.7. Each surface has the same bounding curve, namely, the unit circle in the x-y plane. The left-hand side of Stokes' theorem and, therefore, the right-hand side, is the same for all three surfaces! This highlights the fact that Stokes' theorem is a "fundamental theorem of calculus" for the curl in that the evaluation of the right-hand side of Eq. (7.45) is determined entirely by the nature of the boundary $\partial\sigma$.

Example. Consider the surface σ given by the surface of the upper half-sphere of radius R:

$$\sigma\colon x^2+y^2+z^2=R^2\,,\quad (z\geq 0)\,. \tag{7.46}$$

The bounding curve is therefore given by the circle of radius R in the x-y plane:

$$\partial\sigma\colon x^2+y^2=R^2\,. \tag{7.47}$$

These quantities are shown Fig. 7.7(a). We will evaluate both side Stokes' theorem in Eq. (7.45) for the vector field

$$\boldsymbol{V}=-y\,\boldsymbol{i}+x\,\boldsymbol{j}+z\,\boldsymbol{k}\,. \tag{7.48}$$

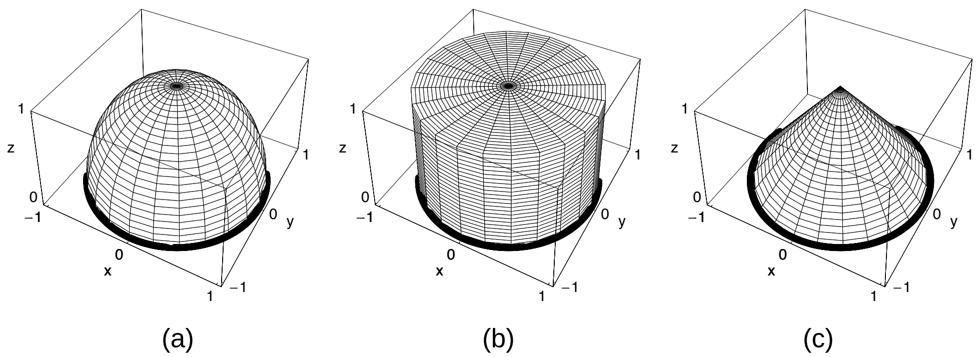

Figure 7.7: Three surfaces that have the same bounding curve in the x-y plane, which is shown emboldened: (a) an upper half-sphere, (b) a cylinder, and (c) a cone. For each of these surfaces and for a given vector field, the left-hand side of Stokes' theorem in Eq. (7.45) is identical.

We consider the left-hand side of Eq. (7.45) first. We have

$$\begin{aligned} \boldsymbol{V} \cdot d\boldsymbol{r} &= (-y\,\boldsymbol{i} + x\,\boldsymbol{j} + z\,\boldsymbol{k}) \cdot (dx\,\boldsymbol{i} + dy\,\boldsymbol{j} + dz\,\boldsymbol{k}) \\ &= -y\,dx + x\,dy + z\,dz\,. \end{aligned} \tag{7.49}$$

On $\partial\sigma$, which lies in the x-y plane, where $z = 0$, this expression reduces to

$$\boldsymbol{V} \cdot d\boldsymbol{r}\Big|_{\partial\sigma} = -y\,dx + x\,dy\,. \tag{7.50}$$

In circular polar coordinates, $x = R\cos\phi$, $y = R\sin\phi$, where $0 \le \phi < 2\pi$, we have

$$dx = -R\sin\phi\,d\phi\,, \qquad dy = R\cos\phi\,d\phi\,, \tag{7.51}$$

from which we obtain

$$\boldsymbol{V} \cdot d\boldsymbol{r}\Big|_{\partial\sigma} = -y\,dx + x\,dy = R^2\sin^2\phi\,d\phi + R^2\cos^2\phi\,d\phi = R^2\,d\phi\,. \tag{7.52}$$

Thus, the left-hand side of Stokes' theorem evaluates to

$$\oint_{\partial\sigma} \boldsymbol{V} \cdot d\boldsymbol{r} = \int_0^{2\pi} R^2\,d\phi = 2\pi R^2\,. \tag{7.53}$$

To evaluate the right-hand side of Stokes' theorem, we first calculate the curl of $\boldsymbol{V}$:

$$\boldsymbol{\nabla}\times\boldsymbol{V} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & z \end{vmatrix} = (1+1)\,\boldsymbol{k} = 2\,\boldsymbol{k}\,. \tag{7.54}$$

The unit normal is given in Eq. (6.43):

$$\boldsymbol{n} = \frac{x}{R}\,\boldsymbol{i} + \frac{y}{R}\,\boldsymbol{j} + \frac{z}{R}\,\boldsymbol{k}\,. \tag{7.55}$$

Thus,

$$(\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{n} = 2\,\boldsymbol{k}\cdot\left(\frac{x}{R}\,\boldsymbol{i} + \frac{y}{R}\,\boldsymbol{j} + \frac{z}{R}\,\boldsymbol{k}\right) = \frac{2z}{R}\,. \tag{7.56}$$

We will evaluate the integral of this quantity over the upper half-sphere in spherical polar coordinates:

$$\begin{aligned} \int_\sigma (\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{n}\,d\sigma &= R^2\int_0^{2\pi}\int_0^{\frac{1}{2}\pi}\sin\theta\left(\frac{2}{R}\,R\cos\theta\right)\,d\theta \\ &= 4\pi R^2 \underbrace{\int_0^{\frac{1}{2}\pi}\sin\theta\cos\theta\,d\theta}_{\frac{1}{2}\sin^2\theta\Big|_0^{\frac{1}{2}\pi} = \frac{1}{2}} \\ &= 2\pi R^2\,. \end{aligned} \tag{7.57}$$

which agrees with Eq. (7.53). ∎

Example. Consider the surface σ given by a cylinder of radius R and height H:

$$\sigma\colon x^2 + y^2 = R^2\,, \quad (0 \le z \le H)\,. \tag{7.58}$$

The bounding curve is again the circle of radius R in the x-y plane:

$$\partial\sigma\colon x^2 + y^2 = R^2\,. \tag{7.59}$$

These quantities are shown Fig. 7.7(b). We again have the vector field

$$\boldsymbol{V} = -y\,\boldsymbol{i} + x\,\boldsymbol{j} + z\,\boldsymbol{k}\,. \tag{7.60}$$

Since the bounding curve and the vector field are the same as in the preceding example, the evaluation of the left-hand side of Stokes' theorem again yields the value $2\pi R^2$. To evaluate the right-hand side of Stokes' theorem, we again have that $\boldsymbol{\nabla}\times\boldsymbol{V} = 2\,\boldsymbol{k}$. The unit normal to the top of the cylinder $\boldsymbol{n} = \boldsymbol{k}$, so for this part of the surface integral we have

$$(\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{n} = 2\,. \tag{7.61}$$

The integral of this quantity over the top of the cylinder is

$$\int_\sigma (\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{n}\,d\sigma = 2\int_0^R r\,dr\int_0^{2\pi} d\phi = 2\pi R^2\,. \tag{7.62}$$

For the integral over the sides of the cylinder, we have that the unit normal is

$$\boldsymbol{n} = \frac{x}{R}\,\boldsymbol{i} + \frac{y}{R}\,\boldsymbol{j}\,, \tag{7.63}$$

and, therefore, we find that

$$(\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{n} = 2\,\boldsymbol{k}\cdot\left(\frac{x}{R}\,\boldsymbol{i} + \frac{y}{R}\,\boldsymbol{j}\right) = 0\,. \tag{7.64}$$

Thus, the integral over the sides of the cylinder vanishes and the total integral over the surface of the cylinder is given by the integral over the top of the cylinder, which is independent of the height of the cylinder, and equal to $2\pi R^2$. ▮

7.4 Summary

This chapter has introduced the curl of a vector field $\boldsymbol{V} = \boldsymbol{V} = P(x,y,z)\,\boldsymbol{i} + Q(x,y,z)\,\boldsymbol{j} + R(x,y,z)\,\boldsymbol{k}$:

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{V} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}\,. \end{aligned} \tag{7.65}$$

The curl represents the circulation density of the vector field and, because of the derivative operation, has an associated Fundamental Theorem of Calculus called Stokes' theorem theorem:

$$\oint_{\partial\sigma} \boldsymbol{V} \cdot d\boldsymbol{r} = \iint_{\sigma} (\boldsymbol{\nabla} \times \boldsymbol{V}) \cdot \boldsymbol{n} \, d\sigma \,, \tag{7.66}$$

for a surface σ with a bounding curve $\partial\sigma$.