# Chapter 6

# The Divergence and the Divergence Theorem

There are two basic types of function that describe physical quantities in higher spatial dimensions: scalar functions of several variables and vector fields. Scalar functions assign a number to every point in a given region, while vector fields assign a vector, i.e. a magnitude and direction, to every point in a region. For example, temperature is a scalar function, but the wind direction and speed on a weather map is an example of a vector field.

In the preceding chapter, we showed that the gradient of a scalar function is a vector field that characterizes the magnitude and direction of the maximum rate of change of that function. By their very nature, vector fields are more difficult to represent and visualize than scalar functions, so operations that characterize properties of vector fields are especially useful. In this chapter, we introduce the "divergence" of a vector field, a suggestively named quantity that quantifies the extent to which the vector field points toward or away (i.e. *diverges*) from a point. The divergence is central quantity in the mathematical formulations of continuum theories such as electromagnetism, fluid mechanics, and elasticity.

## 6.1 Definition of the Divergence

We will work initially in two spatial dimensions; the results obtained can be taken over to three dimensions with minimal procedural changes. Consider

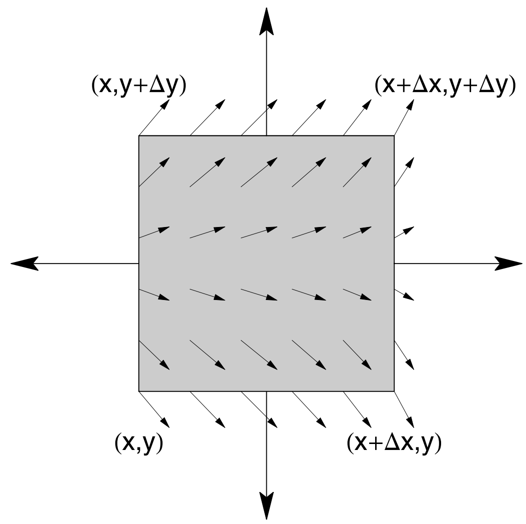


Figure 6.1: The region of area $\Delta x \Delta y$ used to calculated the divergence of a vector field that crosses the boundary of that region. The outward unit normal of each face of this region is also indicated.

the construction in Fig. 6.1, which shows a rectangular region in the $x$-$y$ plane of area $\Delta x \Delta y$. A vector field that crosses the boundary of this region is also indicated. We now ask the question: what is the flux of the vector field across that region? The **flux** of a vector field across a boundary is defined as the part of the vector field that is *normal* to that surface. The component of the vector parallel to the boundary does not contribute to the flux across the boundary. If we denote this vector field by

$$\boldsymbol{V} = P(x,y)\,\boldsymbol{i} + Q(x,y)\,\boldsymbol{j}\,, \tag{6.1}$$

and the outward unit normal of the surface by $\boldsymbol{n}$, the total flux across the boundary of the region can be expressed as a line integral of the "dot" product $\boldsymbol{V} \cdot \boldsymbol{n}$ over the boundary:

$$\int \boldsymbol{V} \cdot \boldsymbol{n}\, ds\,. \tag{6.2}$$

Note the sign convention used here: *positive* for flux *out* of the region and *negative* for flux into the region. We can now calculate the contribution to this quantity from each of the four sides of the rectangular region. Beginning with the part of the boundary contained between $(x, y)$ and $(x + \Delta x, y)$ and moving counterclockwise, we obtain the following expressions:

$$(\boldsymbol{V} \cdot \boldsymbol{n})\Delta x = [P(x,y)\,\boldsymbol{i} + Q(x,y)\,\boldsymbol{j}] \cdot (-\boldsymbol{j})\,\Delta x$$

$$= -Q(x,y)\Delta x\,, \tag{6.3}$$

$$(\boldsymbol{V} \cdot \boldsymbol{n})\Delta y = [P(x+\Delta x,y)\,\boldsymbol{i} + Q(x+\Delta x,y)\,\boldsymbol{j}] \cdot \boldsymbol{i}\,\Delta y$$

$$= P(x+\Delta x,y)\Delta y\,, \tag{6.4}$$

$$(\boldsymbol{V} \cdot \boldsymbol{n})\Delta x = [P(x,y+\Delta y)\,\boldsymbol{i} + Q(x,y+\Delta y)\,\boldsymbol{j}] \cdot \boldsymbol{j}\,\Delta x$$

$$= Q(x,y+\Delta y)\Delta x\,. \tag{6.5}$$

$$(\boldsymbol{V} \cdot \boldsymbol{n})\Delta y = [P(x,y)\,\boldsymbol{i} + Q(x,y)\,\boldsymbol{j}] \cdot (-\boldsymbol{i})\,\Delta y$$

$$= -P(x,y)\Delta y\,, \tag{6.6}$$

where the signs are due to the directions of the unit normals of each face, which are $-\boldsymbol{j}$, $\boldsymbol{i}$, $\boldsymbol{j}$ and $-\boldsymbol{i}$, respectively. Note the arguments of $P$ and $Q$ in each term! The variation along each face has been neglected because the lengths of each side will become infinitesimal later in this calculation. Moreover, the reference points for each side have been chosen so that the leading corrections to $\boldsymbol{V}(x, y)$ are of order $\Delta x \Delta y$, rather than $(\Delta x)^2$ or $(\Delta y)^2$, which would vanish in the infinitesimal limit. Neither of these choices is essential for our calculation, but they make the intermediate step much simpler.

Upon summing up the contributions to the flux, we obtain

$$\int \boldsymbol{V} \cdot \boldsymbol{n}\,ds = [P(x+\Delta x,y) - P(x,y)]\,\Delta y + [Q(x,y+\Delta y) - Q(x,y)]\,\Delta x$$

$$= \left[\frac{P(x+\Delta x,y) - P(x,y)}{\Delta x} + \frac{Q(x,y+\Delta y) - Q(x,y)}{\Delta y}\right] \Delta x \Delta y\,. \tag{6.7}$$

The expressions within the square brackets on the right-hand side are discrete approximations to partial derivatives of $P$ and $Q$, as given in Eqs. (1.12) and

(1.13). Thus, by dividing both sides of this equation by $\Delta x \Delta y$ and taking the limit $\Delta x \to 0$ and $\Delta y \to 0$, we obtain

$$\begin{aligned}
&\lim_{\Delta x\to 0, \Delta y\to 0}\left[\frac{1}{\Delta x \Delta y}\int \boldsymbol{V}\cdot\boldsymbol{n}\,ds\right] \\
&= \lim_{\Delta x\to 0}\left[\frac{P(x+\Delta x, y)-P(x,y)}{\Delta x}\right] + \lim_{\Delta y\to 0}\left[\frac{Q(x, y+\Delta y)-Q(x,y)}{\Delta y}\right] \\
&= \frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\,. \qquad (6.8)
\end{aligned}$$

The quantity on the right-hand side of this equation is called the **divergence** of $\boldsymbol{V}$. From the way we have arrived at this quantity, the divergence is defined as the flux density across the boundary of an infinitesimal region.

This point merits further discussion. We are familiar with other types of densities, such as mass density and charge density. Each of these refers to an extensive quantity in that the total amount of mass or charge within a region is obtained by integrating the corresponding density at each point within the region. The density at any point can be determined by calculating the volume within a region surrounding the point, dividing by the volume of the region, and shrinking the volume of the region to that point. Flux is another such quantity, as the Eq. (6.8) demonstrates. This is of fundamental significance for the applications of the divergence to physical theories.

We can write the divergence of a vector field in a more compact form that has a natural extension to three dimensions by introducing the "del" operation

$$\boldsymbol{\nabla} \equiv \boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}+\boldsymbol{k}\frac{\partial}{\partial z}\,. \qquad (6.9)$$

By regarding this operation as a "vector", we take the "dot" product with

$$\boldsymbol{V} = P(x,y,z)\,\boldsymbol{i} + Q(x,y,z)\,\boldsymbol{j} + R(x,y,z)\,\boldsymbol{k}\,, \qquad (6.10)$$

to obtain

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = \left(\boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}+\boldsymbol{k}\frac{\partial}{\partial z}\right)\cdot(P\,\boldsymbol{i}+Q\,\boldsymbol{j}+R\,\boldsymbol{k}) = \frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\,. \qquad (6.11)$$

The definition of the divergence in Eq. (6.8) can now be written as

$$\boldsymbol{\nabla}\cdot\boldsymbol{V}=\lim_{\Delta A\to 0}\left(\frac{1}{\Delta A}\int \boldsymbol{V}\cdot\boldsymbol{n}\,ds\right), \tag{6.12}$$

where we have written $\Delta A=\Delta x\Delta y$. This definition has several similarities with the definition of the derivative in Eq. (1.1). We used Eq. (1.1) explicitly in arriving at the expression for the divergence, but the similarities run much deeper. The right-hand sides of both equations are expressed in terms of a function evaluated on the boundary of a region [the end-points of an interval in the definition in Eq. (1.1)], and the corresponding derivatives of those functions are obtained by shrinking this region to zero. These similarities will be extended further when we derive the Fundamental Theorem associated with the divergence. We first consider some examples of vector fields and their divergences.

**Example.** As our first example, we consider the vector field

$$\boldsymbol{V}=x\,\boldsymbol{i}+y\,\boldsymbol{j}\,, \tag{6.13}$$

which is shown in a region around the origin in Fig. 6.2 at right. The divergence of $\boldsymbol{V}$ is calculated from the definition in Eq. (6.11), with $P=x$ and $Q=y$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{V}=\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}$$

$$=1+1=2\,. \tag{6.14}$$

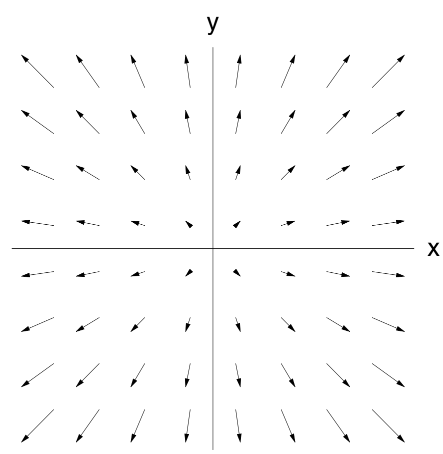

Figure 6.2: Plot of $\boldsymbol{V}=x\,\boldsymbol{i}+y\,\boldsymbol{j}$.

According to our convention, the positive divergence of this vector field indicates that the flux density is directed outward. This is certainly apparent at the origin, but the fact that the divergence is independent of position means that this interpretation is valid for every point in the $x$-$y$ plane, i.e. the flux density is directed outward from *every* point. Understanding how this conclusion is drawn is central to the concept of what the divergence means. The details of this interpretation is discussed in Classwork 7 for this and other vector fields. ▮

**Example.** Consider now the vector field

$$\boldsymbol{V} = x\,\boldsymbol{i} - y\,\boldsymbol{j}\,, \qquad (6.15)$$

which is shown in Fig. 6.3 at right. The divergence of $\boldsymbol{V}$ is calculated from Eq. (6.11), with $P = x$ and $Q = -y$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = \frac{\partial x}{\partial x} - \frac{\partial y}{\partial y}$$

$$= 1 - 1 = 0\,. \qquad (6.16)$$

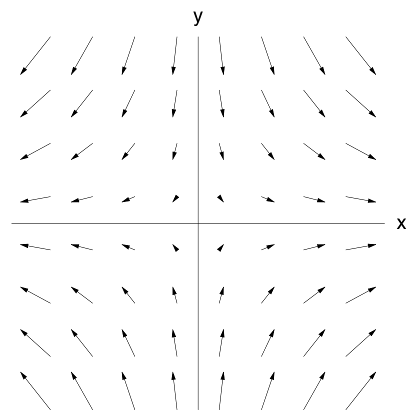


Figure 6.3: Plot of $\boldsymbol{V} = x\,\boldsymbol{i} - y\,\boldsymbol{j}$.

This zero divergence indicates that there is no *net* flux density. This is again apparent at the origin, but the fact that the divergence is independent of position means that this interpretation is valid for any point in the $x$-$y$ plane. ▮

**Example.** Our final example is the vector field

$$\boldsymbol{V} = -y\,\boldsymbol{i} + x\,\boldsymbol{j}\,, \qquad (6.17)$$

which is shown in Fig. 6.4 at right. The divergence of $\boldsymbol{V}$ is calculated from Eq. (6.11), with $P = -y$ and $Q = x$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = -\frac{\partial y}{\partial x} + \frac{\partial x}{\partial y}$$

$$= 0 + 0 = 0\,. \qquad (6.18)$$

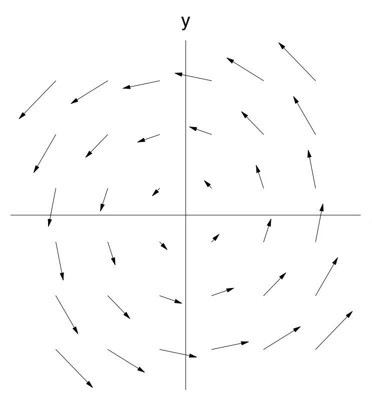


Figure 6.4: Plot of $\boldsymbol{V} = -y\,\boldsymbol{i} + x\,\boldsymbol{j}$.

The vanishing divergence of this vector field indicates that there is no net flux density, but here because the vector appears as a "vortex". As in the preceding examples, this conclusion is valid at *every* point. ▮

## 6.2 The Divergence Theorem

The divergence was derived in the preceding section by calculating the flux through an infinitesimal region in the $x$-$y$ plane. In this section, we will integrate Eq. (6.8) to obtain the "fundamental theorem of calculus" associated with the divergence. The key point here is that for adjacent regions the share a boundary, the flux from one region exactly cancels the flux into the next region. The only *net* contribution to the flux is where there is no adjacent region, as shown in Fig. 6.5. Thus, we can partition any region in the $x$-$y$ plane into contiguous regions of the type in Figs. 6.1 and invoke the cancellation of flux along adjacent boundaries (Fig. 6.5). The sequence of steps is shown in Fig. 6.6. If we denote the boundary of the $i$th region by $\Delta\sigma_i$ and the area $\Delta x\Delta y$ by $\Delta\tau_i$, then for each region we have

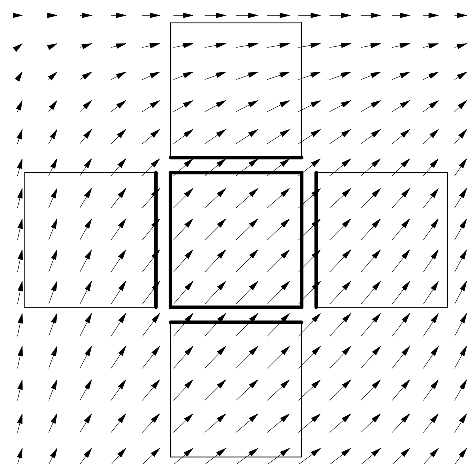

Figure 6.5: Adjacent regions showing that, where there is a common boundary, indicated by bold lines, the flux from one region exactly cancels the flux into the adjacent region. Only unshared boundaries contribute to the net flux.

$$\int_{\Delta\sigma_i} \boldsymbol{V}\cdot\boldsymbol{n}_i\,ds = \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\Delta\tau_i\,. \tag{6.19}$$

where $\boldsymbol{n}_i$ represents the outward normal of the $i$th region. Then, summing over each region yields

$$\sum_i \int_{\Delta\sigma_i} \boldsymbol{V}\cdot\boldsymbol{n}_i\,ds = \sum_i \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\Delta\tau_i\,. \tag{6.20}$$

By taking the limit $\Delta x\to 0$ and $\Delta y\to 0$, we obtain

$$\lim_{\Delta x\to 0,\Delta y\to 0}\left[\sum_i \int_{\Delta\sigma_i} \boldsymbol{V}\cdot\boldsymbol{n}_i\,ds\right] = \int_\sigma \boldsymbol{V}\cdot\boldsymbol{n}\,ds \equiv \int \boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma\,, \tag{6.21}$$

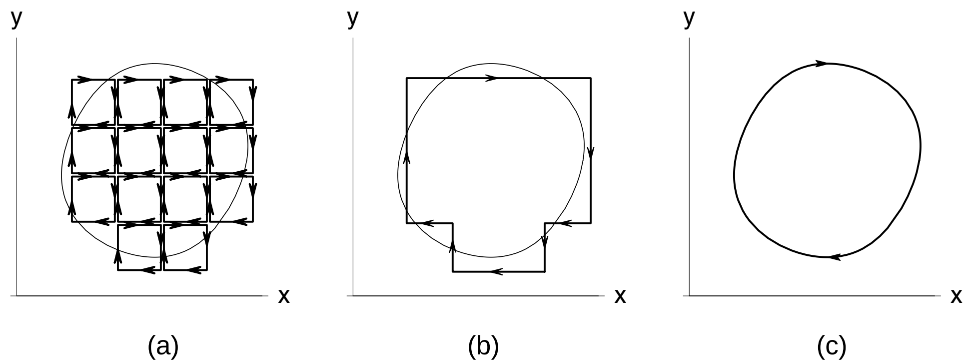


Figure 6.6: (a) Partition of a region in the $x$-$y$ plane bounded by a curve. (b) Cancellation of integrals over adjacent regions, as shown in Fig. 6.5. (c) As the area of the basic regions becomes smaller ($\Delta x \to 0$ and $\Delta y \to 0$), the partitioning in (a) provides a successively more accurate representation of the region, yielding a more accurate representation of the curve surrounding the region.

where $\sigma$ is the boundary of the *entire* region and $\boldsymbol{n}$ is the corresponding outward unit normal, and

$$\lim_{\Delta x\to 0, \Delta y \to 0}\left[\sum_i \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\Delta\tau_i\right] = \iint \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right) d\tau\,. \tag{6.22}$$

Thus,

$$\int \boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma = \iint \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right) d\tau\,, \tag{6.23}$$

which is the **divergence theorem in two dimensions**. The same arguments apply in three dimensions, and we can write this in a more compact form by using the notation in Eq. (6.11):

$$\boxed{\int \boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma = \iint \boldsymbol{\nabla}\cdot\boldsymbol{V}\,d\tau\,,} \tag{6.24}$$

where $\sigma$ is the curve bounding the area $\tau$ in two dimensions, and

$$\boxed{\iint \boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma = \iiint \boldsymbol{\nabla}\cdot\boldsymbol{V}\,d\tau\,,} \tag{6.25}$$

where $\sigma$ is the surface bounding the volume $\tau$ in three dimensions. Note that these equations have the structure of the Fundamental Theorem of Calculus in Eqs. (1.25) and (1.26), which we combine as

$$F(b) - F(a) = \int_a^b \frac{dF}{dx}\, dx\,. \tag{6.26}$$

The left-hand sides of all these equations involves the integral of the derivative of a function over the interior of a region, while the right-hand sides involve the evaluation of the function over the boundary of that region.

**Example.** We will verify the divergence theorem Eq. (6.24) for the vector field

$$\boldsymbol{V} = x\,\boldsymbol{i} + y\,\boldsymbol{j} \tag{6.27}$$

over the volume in the region of the $x$-$y$ plane given by $0 \le x \le 1$ and $0 \le y \le 1$, as depicted in Fig. 6.7. The divergence of $\boldsymbol{V}$ was calculated in the Example in the preceding section, $\boldsymbol{\nabla}\cdot\boldsymbol{V} = 2$, so the right-hand side of the divergence can be evaluated immediately:

$$\iint \boldsymbol{\nabla}\cdot\boldsymbol{V}\, d\tau = 2\int_0^1 dx \int_0^1 dy = 2\,. \tag{6.28}$$

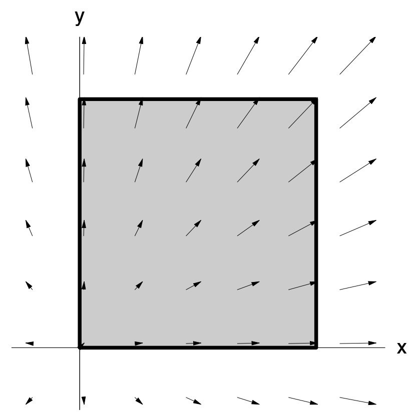


Figure 6.7: Plot of $\boldsymbol{V} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$ in the region $0 \le x \le 1$ and $0 \le y \le 1$ of the $x$-$y$ plane, shown shaded with emboldened boundaries.

The left-hand side is evaluated in an analogous manner to that used to obtain Eq . (6.8). Beginning with the segment along the $x$-axis and proceeding in a counterclockwise direction, we have

$$\boldsymbol{V}\cdot\boldsymbol{n} = (x\,\boldsymbol{i} + y\,\boldsymbol{j})\cdot(-\boldsymbol{j}) = -y \tag{6.29}$$

$$\boldsymbol{V}\cdot\boldsymbol{n} = (x\,\boldsymbol{i} + y\,\boldsymbol{j})\cdot\boldsymbol{i} = x \tag{6.30}$$

$$\boldsymbol{V}\cdot\boldsymbol{n} = (x\,\boldsymbol{i} + y\,\boldsymbol{j})\cdot\boldsymbol{j} = y \tag{6.31}$$

$$\boldsymbol{V}\cdot\boldsymbol{n} = (x\,\boldsymbol{i} + y\,\boldsymbol{j})\cdot(-\boldsymbol{i}) = -x \tag{6.32}$$

Along these four segments, we have, respectively, that $y = 0$, $x = 1$, $y = 1$, and $x = 0$. Thus, only the expressions in Eqs. (6.30) and (6.31) have nonzero contributions. This can be understood from Fig. 6.7 because $\boldsymbol{V}$ is seen to have only components *parallel* to the boundaries that lie along the $x$- and $y$-axes. Hence, there is no flux of $\boldsymbol{V}$ across these boundaries. We thereby obtain

$$\int \boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma = \int_0^1 x\Big|_{x=1} dy + \int_0^1 y\Big|_{y=1} dx = 1 + 1 = 2\,, \tag{6.33}$$

which agrees with Eq. (6.28). ▮

**Example.** We now turn our attention to the divergence theorem in three dimensions in Eq. (6.25). Consider the flux of vector field

$$\boldsymbol{V} = y\,\boldsymbol{i} - x\,\boldsymbol{j} + z\,\boldsymbol{k}\,, \tag{6.34}$$

through the surface of the unit sphere, $x^2 + y^2 + z^2 = 1$ (Fig. 6.8). To evaluate the right-hand side of Eq. (6.25), we first calculate the divergence of $\boldsymbol{V}$. Using Eq. (6.11) with $P = y$, $Q = -x$, and $R = z$, we obtain

$$\begin{aligned}\nabla\cdot\boldsymbol{V} &= \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} + \frac{\partial z}{\partial z}\\ &= 0 + 0 + 1 = 1\,. \end{aligned} \tag{6.35}$$

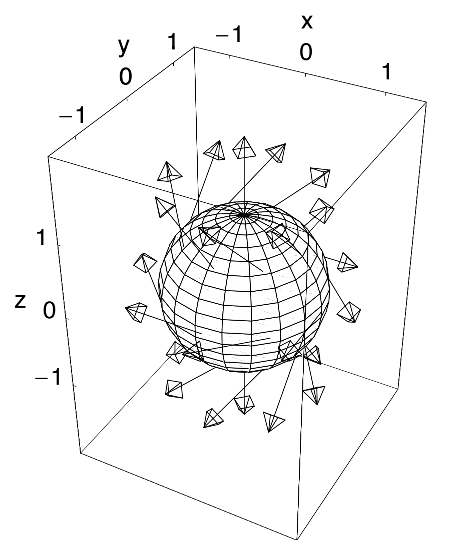

Figure 6.8: The vector field in Eq. (6.34) evaluated at the surface of the unit sphere.

The integral of this quantity over the volume of the sphere can be carried out either by inspection or explicitly in spherical polar coordinates:

$$\iiint \nabla\cdot\boldsymbol{V}\,d\tau = \int_0^1 r^2\,dr \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta = \frac{1}{3}\times 2\pi\times 2 = \frac{4}{3}\pi\,, \tag{6.36}$$

which is just the volume of the unit sphere.

The evaluation of the left-hand side of Eq. (6.25) requires determining the "dot" product $\boldsymbol{V}\cdot\boldsymbol{n}$ over the surface of the unit sphere. The outward unit normal is obtained from the gradient

$$\boldsymbol{\nabla}(x^2+y^2+z^2) = 2x\,\boldsymbol{i} + 2y\,\boldsymbol{j} + 2z\,\boldsymbol{k}\,, \tag{6.37}$$

The length of this vector is

$$|\boldsymbol{\nabla}(x^2+y^2+z^2)| = \sqrt{4x^2+4y^2+4z^2} = 2\sqrt{x^2+y^2+z^2} = 2 \tag{6.38}$$

on the surface of the unit sphere (where $x^2+y^2+z^2=1$). Hence,

$$\boldsymbol{n} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}\,, \tag{6.39}$$

so

$$\boldsymbol{V}\cdot\boldsymbol{n} = (y\,\boldsymbol{i} - x\,\boldsymbol{j} + z\,\boldsymbol{k})\cdot(x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}) = xy - xy + z^2 = z^2\,. \tag{6.40}$$

The integral of this quantity over the surface of the unit sphere is carried out in spherical polar coordinates, with $z^2 = \cos^2\theta$:

$$\iint \boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma = \underbrace{\int_0^{2\pi} d\phi}_{2\pi}\,\underbrace{\int_0^{\pi} \sin\theta\cos^2\theta\,d\theta}_{-\frac{1}{3}\cos^3\theta\Big|_0^{\pi} = \frac{2}{3}} = \frac{4}{3}\pi\,, \tag{6.41}$$

which agrees with Eq. (6.36). ∎

## 6.3 Gauss' Law

Gauss' law is a special case of the divergence theorem that has several important applications in physics. We begin by considering the divergence theorem for the particular case that

$$\boldsymbol{V} = \boldsymbol{\nabla}\Phi(r)\,, \tag{6.42}$$

i.e. $\boldsymbol{V}$ is the gradient of a scalar function $\Phi$ and $r = (x^2+y^2+z^2)^{1/2}$ is the usual radial variable that measures the distance of the point $(x, y, z)$ to the origin. We will evaluate the left-hand side of Eq. (6.25) for a spherical surface of radius $R$, which means that we need to determine $\boldsymbol{V}$ and $\boldsymbol{n}$ over this surface. The equation of this surface is $x^2+y^2+z^2 = R^2$, so the outward unit normal is obtained by taking the gradient of this expression and normalizing the resulting vector, as in the steps leading to Eq. (6.39):

$$\boldsymbol{n} = \frac{x}{R}\,\boldsymbol{i} + \frac{y}{R}\,\boldsymbol{j} + \frac{z}{R}\,\boldsymbol{k}\,. \tag{6.43}$$

To calculate $\boldsymbol{V}$, we use the chain rule to obtain

$$\begin{aligned} \boldsymbol{\nabla}\Phi(r) &= \frac{\partial\Phi}{\partial x}\,\boldsymbol{i} + \frac{\partial\Phi}{\partial y}\,\boldsymbol{j} + \frac{\partial\Phi}{\partial z}\,\boldsymbol{k} \\ &= \frac{d\Phi}{dr}\frac{\partial r}{\partial x}\,\boldsymbol{i} + \frac{d\Phi}{dr}\frac{\partial r}{\partial y}\,\boldsymbol{j} + \frac{d\Phi}{dr}\frac{\partial r}{\partial z}\,\boldsymbol{k} \\ &= \frac{d\Phi}{dr}\left(\frac{x}{r}\,\boldsymbol{i} + \frac{y}{r}\,\boldsymbol{j} + \frac{z}{r}\,\boldsymbol{k}\right). \end{aligned} \tag{6.44}$$

On the surface of the sphere, $r = R$ and this expression reduces to

$$\boldsymbol{\nabla}\Phi(r) = \frac{d\Phi}{dr}\bigg|_{r=R}\left(\frac{x}{R}\,\boldsymbol{i} + \frac{y}{R}\,\boldsymbol{j} + \frac{z}{R}\,\boldsymbol{k}\right), \tag{6.45}$$

so

$$\boldsymbol{V}\cdot\boldsymbol{n} = \boldsymbol{\nabla}\Phi(r)\cdot\boldsymbol{n} = \frac{d\Phi}{dr}\bigg|_{r=R}\left(\frac{x^2+y^2+z^2}{R^2}\right) = \frac{d\Phi}{dr}\bigg|_{r=R}, \tag{6.46}$$

which, since both $\boldsymbol{V}$ and $\boldsymbol{n}$ are radial vector fields, is a *constant* on the surface of the sphere. Thus, integrating this quantity (which is a constant) over the surface of the sphere of radius $R$ yields

$$\iint \boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma = 4\pi R^2\,\frac{d\Phi}{dr}\bigg|_{r=R}. \tag{6.47}$$

We now observe that, if we specialize our choice of $\Phi$ to $\Phi(r) = A/r$, where $A$ is any constant, then

$$\iint \boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma = -4\pi A\,, \tag{6.48}$$

i.e. *independent of the radius of the sphere*!

This result has several important consequences. Figure 6.9 shows a two-dimensional depiction that we will use in the following discussion. Figure

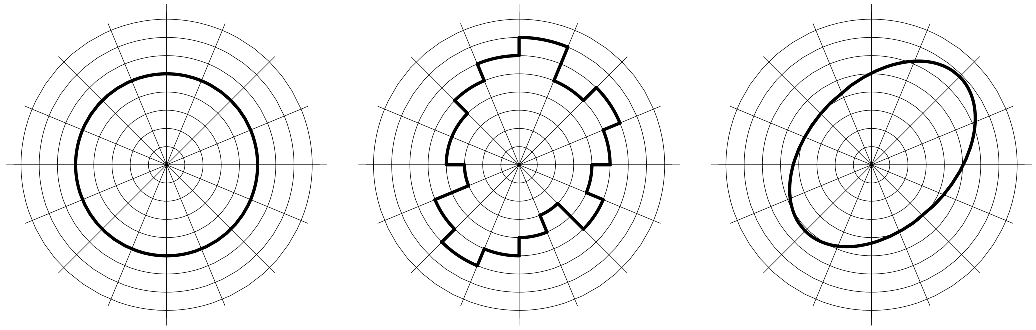


Figure 6.9: Schematic depiction in two dimensions of Gauss' law for surfaces that enclose the origin, with (a) a "spherical surface", (b) a deformed spherical surface, which leaves the value of Eq. (6.48) unaffected, and (c) a general surface that can be represented by the construction in (b).

6.9(a) shows a "sphere" of any radius. Since Eq. (6.48) is independent of the radius, the value of this integral is unaffected by any deformation of this sphere as long as the resulting surface contains the origin. One such deformation is shown in Fig. 6.9. There is no flux through the radial planes because the vector field is radial, so only the spherical portions contribute to the flux. Any section within a fixed subtended angle can be moved to any radius with no effect on the flux through that section. Hence, since any surface can be decomposed into such radial and spherical sections, the result in Eq. (6.48) will be obtained for *any* surface that enclosed the origin, as shown in Fig. 6.9(c).

An altogether different result is obtained if the surface does not contain the origin. This situation is depicted in Fig. 6.10. The surface in Fig. 6.10(a) is composed of spherical sections. Since this surface surrounds a region that excludes the origin, the flux into the volume exactly cancels the flux out of the volume because every spherical section which admits flux has a corresponding region that expels the same amount of flux. Hence, the flux integral over such a surface vanishes! Figure 6.10(b) shows a smooth surface that can be decomposed into sections as in Fig. 6.10(b). Figures 6.9 and 6.10 summarize the essence of Gauss' law.

One of the most far-reaching applications of Gauss' law is to electrostatics, where the function $\Phi(r)$ represents the electrostatic potential of a charge $q$

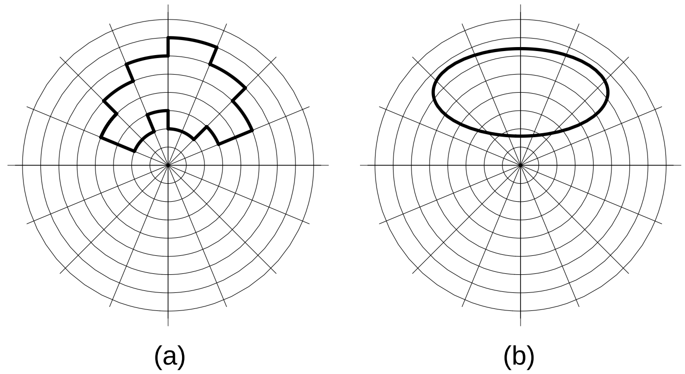


Figure 6.10: Schematic depiction in two dimensions of Gauss' law for surfaces that do not enclose the origin, with (a) a deformed surface with "spherical" sections, (b) a general surface that can be represented by the construction in (a).

located at the origin:

$$\Phi(r) = \frac{q}{4\pi\epsilon_0 r}\,, \tag{6.49}$$

where $\epsilon_0$ is the permittivity of free space. The associated electric field $\boldsymbol{E}$ is then given by

$$\boldsymbol{E} = -\boldsymbol{\nabla}\Phi = \frac{q}{4\pi\epsilon_0 r^2}\,, \tag{6.50}$$

and Gauss' law reads

$$\iiint \boldsymbol{\nabla}\cdot\boldsymbol{E}\,d\tau = \iint \boldsymbol{E}\cdot\boldsymbol{n}\,d\sigma = \frac{q}{\epsilon_0}\,. \tag{6.51}$$

if the surface $\sigma$ encloses the origin. More generally, the right-hand side is equal to the total charge enclosed by $\sigma$, i.e. the total charge contained within the volume $\tau$. Gauss' law results from the divergence theorem applied to Coulomb's law and leads to one of the four Maxwell equations – the equations that govern the behavior of all electromagnetic phenomena.

## 6.4 Summary

This chapter has introduced the divergence of a vector field $\boldsymbol{V} = \boldsymbol{V} = P(x,y,z)\,\boldsymbol{i} + Q(x,y,z)\,\boldsymbol{j} + R(x,y,z)\,\boldsymbol{k}$:

$$\boldsymbol{\nabla} \cdot \boldsymbol{V} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\,. \tag{6.52}$$

The divergence represents the flux density of the vector field and, because of the derivative operation, has an associated Fundamental Theorem of Calculus called the divergence theorem:

$$\iint \boldsymbol{V} \cdot \boldsymbol{n}\, d\sigma = \iiint \boldsymbol{\nabla} \cdot \boldsymbol{V}\, d\tau\,, \tag{6.53}$$

for a surface $\sigma$ surrounding a volume $\tau$.