Chapter 4

Line Integrals

For a function f of a single variable, the integral of f over an interval is uniquely determined once the limits of integration are specified. Extending this construction to the integration of a function f of two or more variables along a path in space connecting specified initial and final points (the limits of integration) leads to entirely new mathematical issues. Foremost among these is that the value of such an integral – called a line integral – generally depends not just on the limits of integration, but on the path that connects these points along which the integration of f is carried out. Thus, the information required to perform a line integral of a given function is comprised of the initial and final points *and* the path connecting them. In this respect line integrals represent a significant conceptual departure from double and triple integrals.

In this chapter, we will first motivate the mathematical structure of a large class of line integrals, using the calculation of work in classical mechanics as a motivation, and work through several examples to demonstrate through explicit calculations that the value of a line integral can depend on the path connecting two points. We will then examine some general properties of line integrals, determine the criterion for the value of a line integral to be independent of the integration path between the limits of integration. Path-dependent and path-independent line integrals each have important applications in several areas of physics, including mechanics, electromagnetic theory, and thermodynamics, which we mention at various places in this chapter.

4.1 Work in Classical Mechanics

A standard calculation in classical mechanics is the work W done by a force $\boldsymbol{F}$ along a path between two points a and b. If the force has a constant magnitude and direction and acts at an angle θ along a path of length $\boldsymbol{r}$, as shown in the figure at right, the work W done by the force is $W = \boldsymbol{F}\cdot\boldsymbol{r}$. Similarly, if $\boldsymbol{F}$ acts only over an infinitesimal distance $d\boldsymbol{r}$, the corresponding work dW done is

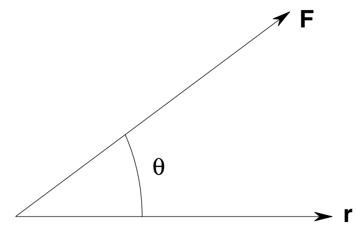

Figure 4.1: A force F acting at an angle θ along a displacement r.

$$dW = \boldsymbol{F}\cdot d\boldsymbol{r}\,. \tag{4.1}$$

Suppose now that the force is a function of position. We consider this situation in one dimension first: $\boldsymbol{F} = \boldsymbol{F}(x)$. The calculation of the work between two points $x = a$ and $x = b$ proceeds according to the construction in Fig. 4.2. The interval (a, b) is first divided into N subintervals of length $\Delta x = (b - a)/N$. The force acting within each of these subintervals is taken

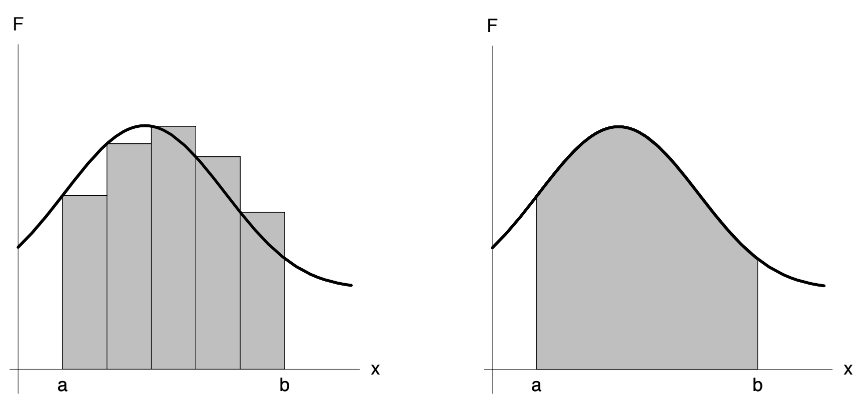

Figure 4.2: (Left panel) Construction used to calculate the work done from $x = a$ to $x = b$ by a position-dependent force. The shaded area corresponds to the work calculated by regarding the force as constant over each subinterval. (Right panel) The corresponding calculation for infinitesimal subintervals, which is seen to represent the area bounded by F, the x-axis, and the lines $x = a$ and $x = b$.

to be the constant value at the left endpoint of that interval. Thus, we obtain

$$F(a)\Delta x + F(a+\Delta x)\Delta x + \cdots + F(b-\Delta x)\Delta x\,. \tag{4.2}$$

as an approximation of the work done over the interval. As $\Delta x \to 0$, this approximation becomes increasingly accurate and the work done approaches the shaded region in the right panel of the figure. Referring to Sec. 1.2, the procedure depicted in Fig. 4.2 is the same as that used for the Riemann sum construction of the integral of a function, so we conclude that

$$\boxed{W = \int_a^b F\,dx\,.} \tag{4.3}$$

Similar considerations apply for paths in two and three dimensions. In this chapter, we will consider the two-dimensional case. A force $\boldsymbol{F}$ in two dimensions is a **vector field**:

$$\boldsymbol{F}(x,y) = P(x,y)\,\boldsymbol{i} + Q(x,y)\,\boldsymbol{j}\,, \tag{4.4}$$

where P and Q are functions of x and y and $\boldsymbol{i}$. This expression indicates that every point (x,y) is assigned a vector $\boldsymbol{F}$ whose x-component is given by $P\,\boldsymbol{i}$ and whose y-component is $Q\,\boldsymbol{j}$. The path along which $\boldsymbol{F}$ acts is a curve $\mathcal{P}$ in the x-y plane between an initial point i and a final point f, as shown in Fig. 4.3. The work done along this path is calculated as in Eq. (4.1) by first considering the incremental work dW done by the force along a distance $d\boldsymbol{r}$: $dW = \boldsymbol{F}\cdot d\boldsymbol{r}$, where $d\boldsymbol{r}$ is the incremental distance along the path. Then, with the position vector given by

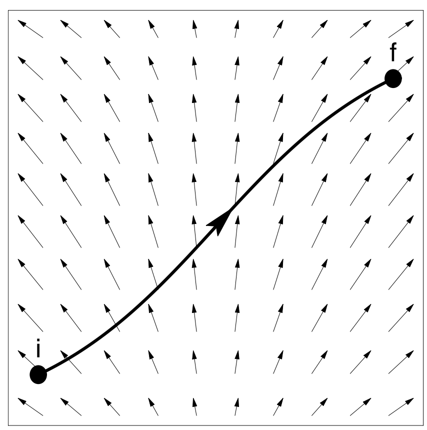

Figure 4.3: A path in a vector field between an initial point i and the final point f.

$$\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j}\,, \tag{4.5}$$

we have that the incremental change along the path is

$$d\boldsymbol{r} = dx\,\boldsymbol{i} + dy\,\boldsymbol{j}\,, \tag{4.6}$$

so the work done along $\mathcal{P}$ is

$$\begin{aligned} W &= \int_{\mathcal{P}} \boldsymbol{F}\cdot d\boldsymbol{r} \\ &= \int_{\mathcal{P}} [P(x,y)\,\boldsymbol{i} + Q(x,y)\,\boldsymbol{j}]\cdot(dx\,\boldsymbol{i} + dy\,\boldsymbol{j}) \\ &= \int_{\mathcal{P}} [P(x,y)\,dx + Q(x,y)\,dy]\ . \end{aligned} \tag{4.7}$$

This is an example of a **line integral**.

In addition to this example from classical mechanics, line integrals appear in thermodynamics and in electricity and magnetism. In thermodynamics, $\mathcal{P}$ represents a *process* between initial and final values of thermodynamic variables (e.g. pressure, temperature, volume). The line integral of such variables yields quantities such as heat flow and the work done during the process. In electricity and magnetism, $\mathcal{P}$ is a path in space, and line integrals represent quantities such as the electromotive force. In all of these cases the mathematical form of a line integral is

$$\boxed{\int_{\mathcal{P}} [f(x,y)\,dx + g(x,y)\,dy]\ ,} \tag{4.8}$$

where f and g are any functions that can be integrated and $\mathcal{P}$ is the path connecting the initial and final points.

As we stressed in the introduction, specifying the integration path $\mathcal{P}$ is as important as specifying the initial and final points. The path provides a functional relationship between x and y and allows the integrals to be evaluated; otherwise the variable y in the term $f(x,y)\,dx$ and the variable x in the term $g(x,y))\,dy$ appear superfluous. Additionally, the value of the line integral may depend explicitly on the path, so specifying only the initial and final points does *not* necessarily sufficient to obtain a unique value. The following example illustrates these ideas.

Example. Consider the line integral

$$\int_{\mathcal{P}} xy\,dx\,, \tag{4.9}$$

which is of the general form in Eq. (4.8) with $f = xy$ and $g = 0$. In the context of the calculation of work, this corresponds to a force $\boldsymbol{F}(x, y) = xy\,\boldsymbol{i}$.

We will evaluate this integral over the three paths shown in Fig. 4.4, each of which have their initial point at the origin $(0, 0)$ and their final point at $(1, 1)$.

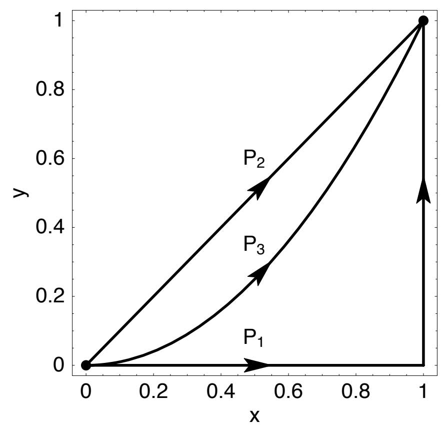

Figure 4.4: The three paths, labelled $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ between $(0, 0)$ and $(1, 1)$ used for evaluating the line integral in Eq.(4.9).

We first consider $\mathcal{P}_1$. This path is composed of two straight segments: $(0, 0) \to (1, 0)$ and $(1, 0) \to (1, 1)$. The first segment lies along the x-axis, so we have that

$$y = 0\,, \qquad 0 \le x \le 1\,. \tag{4.10}$$

Hence, since $y = 0$, the integrand vanishes, so the contribution from segment also vanishes. The second segment is parallel to the y-axis, so

$$x = 1\,, \qquad dx = 0\,, \qquad 0 \le y \le 1\,. \tag{4.11}$$

Since $dx = 0$, the contribution along this segment also vanishes. Therefore, the integral along $\mathcal{P}_1$ vanishes:

$$\int_{\mathcal{P}_1} xy\,dx = 0\,. \tag{4.12}$$

The path $\mathcal{P}_2$ connects $(0, 0)$ to $(1, 1)$ with the straight line $y = x$. Thus, along this path, the integrand can be expressed entirely as a function of x: $xy = x^2$, with $0 \le x \le 1$. The line integral is thereby evaluated as

$$\int_{\mathcal{P}_2} xy\,dx = \int_0^1 x^2\,dx = \tfrac{1}{3}x^3\Big|_0^1 = \tfrac{1}{3}\,. \tag{4.13}$$

Finally, the path $\mathcal{P}_3$ connects $(0, 0)$ to $(1, 1)$ with the parabola $y = x^2$. Along this path, the integrand can be written as $xy = x^3$, with $0 \le x \le 1$, and the line integral becomes

$$\int_{\mathcal{P}_3} xy\,dx = \int_0^1 x^3\,dx = \tfrac{1}{4}x^4\Big|_0^1 = \tfrac{1}{4}\,. \tag{4.14}$$

We have thus obtained three different values for the line integral in Eq. (4.9) along the three paths shown in Fig. 4.4. This result can be understood by interpreting this integral as the work done by the force $\boldsymbol{F} = xy\,\boldsymbol{i}$ over the three paths:

$$\int_{\mathcal{P}_i} \boldsymbol{F}\cdot d\boldsymbol{r} = \int_{\mathcal{P}_i} xy\,dx\,, \tag{4.15}$$

for $i = 1, 2, 3$. This vector field is shown in Fig. 4.5 superimposed on the paths $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$. We can see immediately from this diagram that the work done along $\mathcal{P}_1$ *must* vanish because $\boldsymbol{F}$ vanishes along the x-axis (the first segment of $\mathcal{P}_1$), and acts in the normal direction to the second segment of this path. Alternatively, the line integrals along $\mathcal{P}_2$ and $\mathcal{P}_3$ are both necessarily *positive* because the projection of $\boldsymbol{F}$ onto the path has a component along the direction of the path, producing positive work. ∎

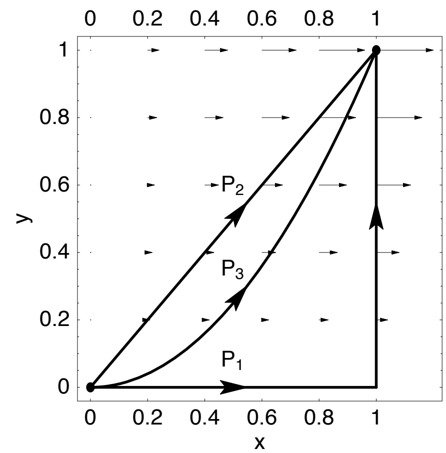

Figure 4.5: The vector field $\boldsymbol{F} = xy\,\boldsymbol{i}$ and the paths $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ shown in Fig. 4.4 used for the evaluation of the line integral in Eq. (4.9).

This example illustrates two fundamental points about line integrals. (i) The value of a line integral may depend on the path over which it is evaluated. There are physical manifestations of this property that have important consequences in mechanics, thermodynamics, and electricity and magnetism. (ii) The path between given initial and final points establishes a relationship between the independent variables. Once this information is incorporated into the line integral, the evaluation reduces to that of an ordinary integral (Sec. 1.2).

Example. Consider the line integral

$$\int_{\mathcal{P}} (xy^2\,dx + x^2y\,dy) \tag{4.16}$$

evaluated along the three paths in Fig. 4.4. Along the first segment of $\mathcal{P}_1$, $y = 0$, and therefore $dy = 0$, so there is no contribution from either term in the integral. Along the second segment $x = 1$, $dx = 0$, and $0 \le y \le 1$. Thus, only the second term in the integral makes a contribution to the integral, and we obtain

$$\int_{\mathcal{P}_1} (xy^2\,dx + x^2y\,dy) = \int_0^1 y\,dy = \tfrac{1}{2}y^2\Big|_0^1 = \tfrac{1}{2}\,. \tag{4.17}$$

Along $\mathcal{P}_2$, $y = x$, so $dy = dx$. Thus, both terms in the integrand can be written in terms of either x or y alone:

$$\int_{\mathcal{P}_2} (xy^2\,dx + x^2y\,dy) = 2\int_0^1 x^3\,dx = 2 \times \tfrac{1}{4}x^4\Big|_0^1 = \tfrac{1}{2}\,, \tag{4.18}$$

which is the same value obtained in Eq. (4.17).

Finally, along $\mathcal{P}_3$, $y = x^2$, so $dy = 2x\,dx$. We can express the integrand in terms of x alone to obtain

$$\begin{aligned}\int_{\mathcal{P}_3} (xy^2\,dx + x^2y\,dy) &= \int_0^1 (x^5\,dx + 2x^5\,dx)\\ &= 3\int_0^1 x^5\,dx = 3 \times \tfrac{1}{6}x^6\Big|_0^1 = \tfrac{1}{2}\,,\end{aligned} \tag{4.19}$$

which is the same as that obtained for the other two paths. A natural question arises: Is this a coincidence, or does this integral always have the same value when evaluated over different paths between fixed initial and final points? The results we have obtained in this example are certainly suggestive, but to address this question in a mathematically concise framework, we must derive some additional properties of line integrals. This is the subject of the next two sections. ▮

4.2 Line Integrals over Closed Curves

An important class of line integrals is that for which the path of integration forms a simple closed curve, i.e. a path that returns to the initial point but does not cross its path. Such integrals find many applications in thermodynamics, where they are called "cycles", and in electricity and magnetism, where they form the mathematical expression of Ampère's law, and are referred to as "loop integrals". In this section, we will re-express the question of the path-dependence of a line integral in terms of the value of that integral around a closed curve. We first determine the effect that reversing the sense of the integration path has on the value of a line integral.

Consider a line integral over a path between an initial point i and a final point f, as shown in Fig. 4.6(a):

$$\int_{\mathcal{P}} (f\,dx + g\,dy)\,. \tag{4.20}$$

Suppose that this path is reversed, so that the new initial point is f and the new final point is i, as shown Fig. 4.6(b). We signify this path by $-\mathcal{P}$ and write the corresponding line integral as

$$\int_{-\mathcal{P}} (f\,dx + g\,dy)\,. \tag{4.21}$$

The relationship between the values of these two line integrals is straightforward to understand. As the examples in the preceding section show, the evaluation of a line integral always reduces to an ordinary integral. Thus, reversing the integration path in a line integral has the effect of interchanging the upper and lower limits of integration. According to the Fundamental Theorem of Calculus, this changes the *sign* of the integral [Eq. (1.30)]. Thus, the line integrals in Eqs. (4.20) and (4.21) have the same absolute value, but opposite signs:

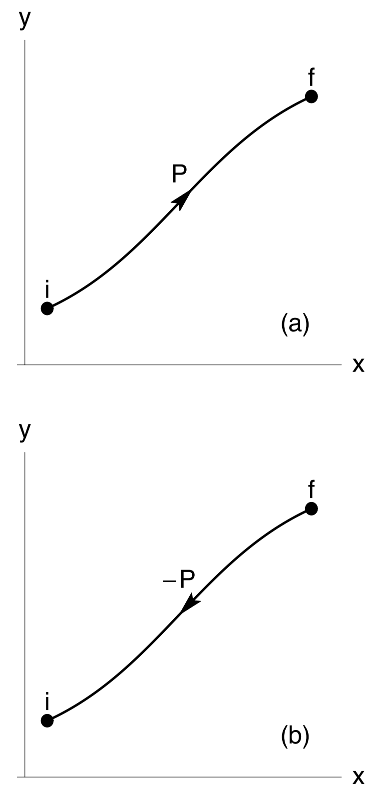

Figure 4.6: (a) The path $\mathcal{P}$ between points i and f, and (b) the reverse path $-\mathcal{P}$ (b).

$$\boxed{\int_{-\mathcal{P}} (f\,dx + g\,dy) = -\int_{\mathcal{P}} (f\,dx + g\,dy)\,.} \tag{4.22}$$

Consider now a line integral over a *closed curve* $\mathcal{C}$ (Fig. 4.7). Such integrals, often called "loop integrals", have a special notation to indicate that the integration path is a closed curve:

$$\oint_{\mathcal{C}} (f\,dx + g\,dy)\,. \tag{4.23}$$

Choose any two distinct points A and B on $\mathcal{C}$ and denote by $\mathcal{P}_1$ the path on $\mathcal{C}$ from A to B and by $\mathcal{P}_2$ the path that returns B to A along $\mathcal{C}$. The integral over $\mathcal{C}$ can be expressed as sum of line integrals over $\mathcal{P}_1$ and $\mathcal{P}_2$:

$$\oint_{\mathcal{C}} (f\,dx + g\,dy) = \int_{\mathcal{P}_1} (f\,dx + g\,dy) + \int_{\mathcal{P}_2} (f\,dx + g\,dy)\,.$$

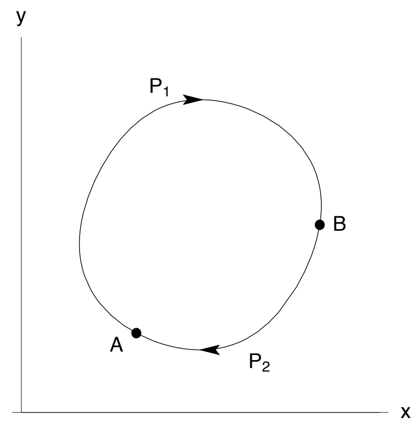

Figure 4.7: A closed curve $\mathcal{C}$ in the x-y plane. $\mathcal{P}_1$ is a path between any two points A and B on $\mathcal{C}$ and $\mathcal{P}_2$ is the path from B to A that completes the loop. The closed curve is the sum of these two paths: $\mathcal{C} = \mathcal{P}_1 + \mathcal{P}_2$.

Suppose that the value of the line integral in Eq. (4.20) is independent of the path $\mathcal{P}$ for any initial and final points. The closed curve $\mathcal{C}$ in Fig. 4.7 defines two paths from A to B: the path $\mathcal{P}_1$ and the *reverse* of the path $\mathcal{P}_2$. Path-independence requires that the line integrals over $\mathcal{P}_1$ and $-\mathcal{P}_2$ are equal:

$$\int_{\mathcal{P}_1} (f\,dx + g\,dy) = \int_{-\mathcal{P}_2} (f\,dx + g\,dy)\,. \tag{4.24}$$

By invoking Eq. (4.22), we can write this equation as

$$\begin{aligned} &\int_{\mathcal{P}_1} (f\,dx + g\,dy) - \int_{-\mathcal{P}_2} (f\,dx + g\,dy) \\ &= \int_{\mathcal{P}_1} (f\,dx + g\,dy) + \int_{\mathcal{P}_2} (f\,dx + g\,dy) \\ &= \oint_{\mathcal{C}} (f\,dx + g\,dy) = 0\,. \end{aligned} \tag{4.25}$$

This shows that, if the value of a line integral is independent of the path between any initial and final points, the loop integral vanishes for any closed curve.

The converse of this statement is also true. If a loop integral vanishes for any closed curve $\mathcal{C}$, then we can choose any two points A and B on $\mathcal{C}$ as initial and final points of line integrals along the corresponding paths $\mathcal{P}_1$ and $\mathcal{P}_2$. Then, by reversing the steps leading to Eq. (4.25), we find that

$$\int_{\mathcal{P}_1}(f\,dx + g\,dy) = \int_{-\mathcal{P}_2}(f\,dx + g\,dy)\,, \tag{4.26}$$

which implies path independence. Thus, we have shown that the path independence of a line integral is both necessary [Eq. (4.26)] and sufficient [Eq. (4.25)] for the loop integral to vanish over any closed curve. In other words, these two properties are *equivalent*:

A line integral

$$\int_{\mathcal{P}}(f\,dx + g\,dy)$$

is independent of the path $\mathcal{P}$ between any two points i and f if and only if

$$\oint_{\mathcal{C}}(f\,dx + g\,dy) = 0$$

for any closed curve $\mathcal{C}$.

This result provides an alternative statement of the fact that line integrals fall into two classes: (i) path-dependent and, therefore, typically non-vanishing values over closed curves, and (ii) path-independent and vanishing values over closed curves. Both types of line integral are important in applications to physics and understanding the physical circumstances that lead to one type of integral or another is a central theme in several disciplines. We conclude this section with two examples.

Example. Consider the loop integral

$$\oint_{\mathcal{C}} y\,dx\,, \tag{4.27}$$

where $\mathcal{C}$ is a circle of radius r centered at $(1,1)$, as shown in Fig. 4.8. We represent the circle as follows:

$$\begin{aligned} x &= 1 - r\cos\phi\,, \\ y &= 1 + r\sin\phi\,, \end{aligned} \tag{4.28}$$

where $0 \le \phi < 2\pi$. This parametrization sweeps through the circle in a clockwise direction beginning at $(1-a, 1)$. The integral in Eq. (4.27) can be expressed as an integral over ϕ by using Eq. (4.28) to transform the integrand, the integration element, and the limits of integration. The integrand y is given by the second of Eqs. (4.28), an application of the chain rule to $x(\phi)$ yields

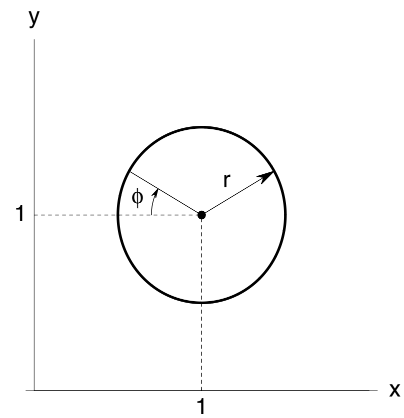

Figure 4.8: The circle of radius a centered at $(1,1)$, showing the definition of ϕ for carrying out a loop integral over this curve.

$$dx = a\sin\phi\,d\phi\,, \tag{4.29}$$

and the limits of integration are $0 \le \phi < 2\pi$. The original integral thereby becomes

$$\begin{aligned} \oint y\,dx &= \int_0^{2\pi} (1 + r\sin\phi) r\sin\phi\,d\phi \\ &= r\underbrace{\int_0^{2\pi} \sin\phi\,d\phi}_{=0} + r^2 \underbrace{\int_0^{2\pi} \sin^2\phi\,d\phi}_{=\pi} \\ &= \pi r^2\,. \end{aligned} \tag{4.30}$$

This is readily identified as the area of the circle enclosed by $\mathcal{C}$. ∎

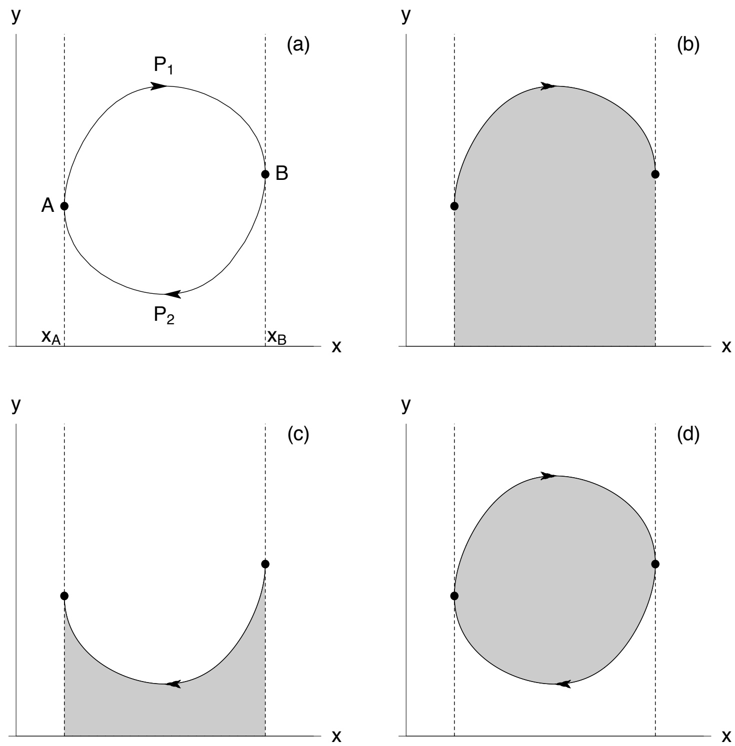

Figure 4.9: The evaluation of the loop integral in Eq. (4.27) around an arbitrary closed curve $\mathcal{C}$, showing (a) the separation of $\mathcal{C}$ into upper and lower paths $\mathcal{P}_1$ and $\mathcal{P}_2$, (b) and (c) the evaluation of the integral along these paths, and (d) the cumulative effect of the loop integral.

This result can be generalized to any closed curve in the x-y plane by following the steps shown in Fig. 4.9. We first identify the points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ that allow $\mathcal{C}$ to be written as the sum of upper and lower paths $\mathcal{P}_2$ and $\mathcal{P}_2$, which can be represented as functions $y_1(x)$ and $y_2(x)$, respectively. The integral along $\mathcal{P}_1$ is

$$\int_{\mathcal{P}_1} y\,dx = \int_{x_A}^{x_B} y_1(x)\,dx\,. \tag{4.31}$$

This is an ordinary integral whose value is represented by area bounded by $y_1(x)$, the x-axis, and $x = x_A$ and $x = x_B$, as shown in Fig. 4.9(a). The loop is completed by integrating $y_2(x)$ from x_B to x_A. The integral

$$\int_{x_A}^{x_B} y_2(x)\,dx \tag{4.32}$$

is represented by the area shown in Fig. 4.9(c). But the integral we need to complete $\mathcal{C}$ has the upper and lower limits interchanged, so its value corresponds to the *negative* of this quantity. Hence, the loop integral is calculated as

$$\oint_{\mathcal{C}} y\,dx = \int_{x_A}^{x_B} y_1(x)\,dx - \int_{x_A}^{x_B} y_2(x)\,dx\,, \tag{4.33}$$

which is represented in Fig. 4.9(d). The integral over $y_2(x)$ cancels the contribution from the integral over $y_1(x)$ that represents the area below $\mathcal{P}_2$, leaving only the area enclosed by $\mathcal{C}$. We have thereby shown that

$$\oint_{\mathcal{C}} y\,dx = A\,, \tag{4.34}$$

where A is the area enclosed by $\mathcal{C}$.

We conclude this section with an example of a loop integral that *does* vanish.

Example. Consider the integral

$$\oint_{\mathcal{C}} (xy^2\,dx + x^2y\,dy)\,, \tag{4.35}$$

where $\mathcal{C}$ is the closed curve in Fig. 4.10. The integrand is the same as that in the second example in Sec. 4.1. The closed curve is composed of four straight segments, so we will evaluate the loop integral by considering each segment separately. Beginning at $(-1,-1)$, the segments are characterized as follows:

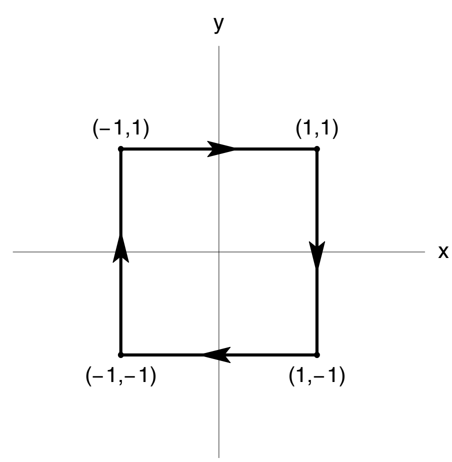

Figure 4.10: The closed contour for the integral in Eq. (4.35).

$$\begin{array}{rcrrrr}
(-1,-1) & \to & (-1,1): & x=-1 & dx=0 & -1\le y\le 1\\
(-1,1) & \to & (1,1): & y=1 & dy=0 & -1\le x\le 1\\
(1,1) & \to & (1,-1): & x=1 & dx=0 & -1\le y\le 1\\
(1,-1) & \to & (-1,-1): & y=-1 & dy=0 & -1\le x\le 1
\end{array}$$

If $dx = 0$ the first term in Eq. (4.35) makes no contribution, while if $dy = 0$, the second term makes no contribution. The integral can therefore be written as (note the upper and lower limits of each integral!)

$$\int_{-1}^{1} y\,dy + \int_{-1}^{1} x\,dx + \int_{1}^{-1} y\,dy + \int_{1}^{-1} x\,dx = 0\,, \tag{4.36}$$

because of the pair-wise cancellation of the integrals. ▮

4.3 Exact and Inexact Differentials

Although the results of the preceding section allow us to re-express the path-independence of a line integral in terms of a loop integral, we are no closer to determining *a priori* whether or not a given line integral is independent of path between given initial and final points. In this section, we will derive a condition that allows us to address this question without having to perform any integration whatsoever.

Consider the line integral

$$\int_{\mathcal{P}} (f\,dx + g\,dy) \tag{4.37}$$

over a path $\mathcal{P}$ between an initial point (x_i, y_i) and a final point (x_f, y_f). The path establishes a relation between x and y that we represent as $y(x)$. This enables us to write the line integral as an integral over x only by following the procedure in Sec. 1.4. We have

$$\int_{\mathcal{P}} f(x,y)\,dx = \int_{x_i}^{x_f} f[x, y(x)]\,dx\,, \tag{4.38}$$

$$\int_{\mathcal{P}} g(x,y)\,dy = \int_{x_i}^{x_f} g[x, y(x))]\,\frac{dy}{dx}\,dx\,. \tag{4.39}$$

Thus,

$$\int_{\mathcal{P}} (f\,dx + g\,dy) = \int_{x_i}^{x_f} \left\{ f[x, y(x)] + g[x, y(x)]\,\frac{dy}{dx} \right\} dx\,. \tag{4.40}$$

The right-hand side of this equation is an ordinary integral over $x_i \le x \le x_f$. Accordingly, if the line integral is path-independent, we can use the

Fundamental Theorem of Calculus to write

$$\int_{x_i}^{x_f} \left\{ f[x, y(x)] + g[x, y(x)] \frac{dy}{dx} \right\} dx = F(x_f) - F(x_i) \,, \tag{4.41}$$

where

$$\frac{dF}{dx} = f[x, y(x)] + g[x, y(x)] \frac{dy}{dx} \,. \tag{4.42}$$

By writing F as $F[x, y(x)]$, we also have

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} \,, \tag{4.43}$$

from which we identify

$$\frac{\partial F}{\partial x} = f \,, \qquad \frac{\partial F}{\partial y} = g \,. \tag{4.44}$$

The quantity F is called the **potential**. On account of Eq. (4.44) we can write the differential of F as

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} y = f\,dx + g\,dy \,, \tag{4.45}$$

in which case the quantity on the right-hand side is independent of the path. This is called an **exact differential**. Otherwise, the quantity $f\,dx + g\,dy$ is called an **inexact differential** and the corresponding line integral is path-dependent. Hence, a line integral of an exact differential can be represented as

$$\int_{\mathcal{P}} (f\,dx + g\,dy) = \int_{\mathcal{P}} dF \,, \tag{4.46}$$

in which case we have that

$$\begin{aligned} \int_{\mathcal{P}} (f\,dx + g\,dy) &= F[x_f, y(x_f)] - F[x_i, y(x_i)] \\ &= F(x_f, y_f) - F(x_i, y_i) \,, \end{aligned} \tag{4.47}$$

In terms of our original formulation in Sec. 4.1, this equation states the work done between and initial point i and a final point f is equal to the change in the potential F.

Equation (4.44) provides a method of testing for the exactness of a differential. By differentiating the first of these equations with respect to y,

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial f}{\partial y}\,, \tag{4.48}$$

the second with respect to x,

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial g}{\partial x}\,, \tag{4.49}$$

and equating the mixed second partial derivatives of F: $F_{yx} = F_{xy}$, we obtain

$$\boxed{\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}\,.} \tag{4.50}$$

The discussion leading to this equation shows that it is a *necessary* condition for a differential to be exact. The procedure described in Problem 2, Problem Set 2 shows that this is also a *sufficient* condition for exactness, thus demonstrating the *equivalence* between Eq. (4.50) and the exactness of a differential.

Example. Consider the line integral

$$\int_{\mathcal{P}} y\,dx\,, \tag{4.51}$$

which was discussed in an earlier Example in this section. In the notation of Eq. (4.8), $f = y$ and $g = 0$. Thus,

$$\frac{\partial f}{\partial y} = 1\,, \qquad \frac{\partial g}{\partial x} = 0\,. \tag{4.52}$$

Since these two partial derivatives are unequal, we conclude from Eq. (4.50) that $y\,dx$ is an *inexact differential*, so the line integral in Eq. (4.51) is *path-dependent*. This result is to be expected in view of Eq. (4.34).

As a second example, we consider the line integral

$$\int_{\mathcal{P}} (xy^2\,dx + x^2 y\,dy)\,. \tag{4.53}$$

This integral has been discussed in Secs. 4.1 and 4.2. In the notation of Eq. (4.8), $f = xy^2$ and $g = x^2y$, and we find

$$\frac{\partial f}{\partial y} = 2xy \,, \qquad \frac{\partial g}{\partial x} = 2xy \,. \tag{4.54}$$

The equality of these partial derivatives means that $xy^2\,dx + x^2y\,dy$ is an *exact differential*, so the line integral in Eq. (4.53) is *path-dependent*. This conclusion confirms our expectations based on the results already obtained for this line integral.

The exactness of this differential implies that there is an underlying potential function F such that

$$\frac{\partial F}{\partial x} = xy^2 \,, \qquad \frac{\partial F}{\partial y} = x^2y \,. \tag{4.55}$$

Integrating the first of these with respect to x yields

$$F(x,y) = \tfrac{1}{2}x^2y^2 + h(y) \,, \tag{4.56}$$

where $h(y)$ is an arbitrary function of y (analogous to constants of integration obtained when integrating functions of one variable). Differentiating both sides of this equation with respect to y,

$$\frac{\partial F}{\partial y} = x^2y + h'(y) \,, \tag{4.57}$$

and requiring that this result be consistent with the second of equations (4.55), necessitates setting $h =$ constant, so $h'(y) = 0$. Thus,

$$F(x,y) = \tfrac{1}{2}x^2y^2 + \text{constant} \,. \tag{4.58}$$

The constant term disappears upon integration:

$$\int_{\mathcal{P}} (xy^2\,dx + x^2y\,dy) = \tfrac{1}{2}(x_f^2y_f^2 - x_i^2y_i^2) \,. \tag{4.59}$$

∎

4.4 Arc Length*

A line integral with a mathematical structure different from that in Eq. (4.8) is obtained by calculating the distance travelled by a particle moving along a trajectory. The trajectory is described by a curve $[x(t), y(t)]$, where t is the time between an initial time t_i and a final time t_f. The distance travelled by the particle in time dt, when x changes by dx and y changes by dy, is given by the relation

$$ds^2 = dx^2 + dy^2\,, \tag{4.60}$$

or,

$$ds = (dx^2 + dy^2)^{1/2}\,. \tag{4.61}$$

Thus, the total distance S travelled by the particle, called the **arc length** of the path, is

$$S = \int_i^f ds = \int_i^f \sqrt{dx^2 + dy^2}\,. \tag{4.62}$$

The integrand on the right-hand side can be written in a more physically suggestive form as

$$\sqrt{dx^2 + dy^2} = dt\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/2} = dt\sqrt{v_x^2 + v_y^2} = v\,dt\,, \tag{4.63}$$

where v_x and v_y are x and y components of the instantaneous speed v of the particle. The arc length along the trajectory is can thereby be represented as

$$S = \int_{t_i}^{t_f} v(t)\,dt\,. \tag{4.64}$$

The general form of the arc length,

$$\boxed{\int_{\mathcal{P}} \sqrt{dx^2 + dy^2}\,.} \tag{4.65}$$

is used to represent the distance along any curve $\mathcal{P}$.

Example. We will illustrate the methodology of computing the arc length by considering $y = \cosh x$ between $x = 0$ and $x = a$. For $y = \cosh x$, we have

$$dy = \sinh x\,dx\,, \tag{4.66}$$

so the integrand in Eq. (4.65) becomes

$$\sqrt{dx^2 + dy^2} = \sqrt{dx^2 + \sinh^2 x\, dx} = dx\sqrt{1 + \sinh^2} = \cosh x\, dx\,. \qquad (4.67)$$

Thus,

$$\int \sqrt{dx^2 + dy^2} = \int_0^a \cosh x\, dx = \sinh x\Big|_0^a = \sinh a\,. \qquad (4.68)$$

∎

4.5 Summary

We can summarize the main results we have obtained on line integrals by noting that the following statements are equivalent in that any one implies any other. If any one statement is false, all other are false as well.

1. $f\,dx + g\,dy$ is an exact differential;

2. $\displaystyle\int_{\mathcal{P}} (f\,dx + g\,dy)$ is independent of the path $\mathcal{P}$ between fixed endpoints;

3. $\displaystyle\oint_{\mathcal{C}} (f\,dx + g\,dy) = 0$ for any closed curve $\mathcal{C}$;

4. $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$;

5. There is a potential function F such that $F_x = f$ and $F_y = g$, so $dF = f\,dx + g\,dy$;

6. $\displaystyle\int_{\mathcal{P}} (f\,dx + g\,dy) = F(x_f, y_f) - F(x_i, y_i)$ for any initial point (x_i, y_i) and final point (x_f, y_f).