Chapter 3

Triple Integrals

Triple integrals are the natural extensions of double integrals to three dimensions. The basic physical motivation of such integrals is the same as for double integrals: determining the amount of a quantity, typically expressed as a density, within a three-dimensional region necessitates performing a triple integral of the quantity over that region. Just as for double integrals, there are coordinate systems other than Cartesian that are convenient for integrating over certain types of regions. We will discuss the two most common of such coordinate systems, circular cylindrical coordinates and spherical polar coordinates, and show how integrals are transformed into these coordinate systems.

3.1 Integrals in Cartesian Coordinates

Suppose there is a quantity f that represents the density of a physical quantity, such as the mass or charge density at every point (x, y, z) in a region of three-dimensional space. The amount of this quantity within a region V is obtained by integrating over the ranges of x, y, and z that span the interior of V. Since this calculation involves three separate integrations, it is called a **triple integral**, and is written as

$$\boxed{\iiint_V f(x, y, z)\, dx\, dy\, dz\,.} \tag{3.1}$$

Following our discussion of double integrals, there are several points to note about triple integrals:

1. Once the volume V has been specified, the integral has a unique value.
2. The integrals over x, y, and z can be carried out in any order.
3. If $f = 1$, the integral yields the volume of the integration region:
$$\iiint_V dx\, dy\, dz = V\,. \tag{3.2}$$

The evaluation of triple integrals proceeds in direct analogy to the cases described in Chapter 3 for double integrals. The following examples illustrate the different situations that can arise.

Example. Suppose $f = xyz$, and that V is the volume shown in Fig. 3.1. We must first determine the ranges of the integration variables. The volume V is a cube in the positive octant of space with one corner at the origin. The points (x, y, z) within the cube have coordinates within the ranges

$$\begin{aligned} 0 &\le x \le 1\,, \\ 0 &\le y \le 1\,, \\ 0 &\le z \le 1\,. \end{aligned} \tag{3.3}$$

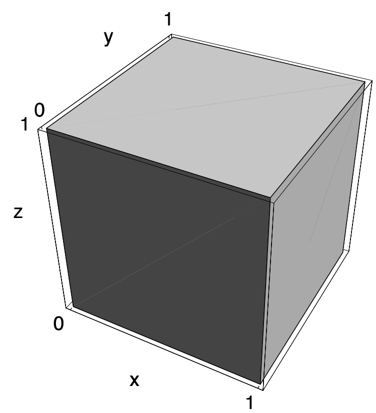

Figure 3.1: The cubic region for the triple integral in Eq. (3.4).

The triple integral of $f = xyz$ within the cube in Fig. 3.1 is therefore calculated as

$$\iiint_V f(x,y,z)dx\, dy\, dz = \underbrace{\int_0^1 x\, dx}_{\frac{1}{2}} \underbrace{\int_0^1 y\, dy}_{\frac{1}{2}} \underbrace{\int_0^1 z\, dz}_{\frac{1}{2}} = \frac{1}{8}\,. \tag{3.4}$$

▮

This type of region, where the ranges of x, y, and z are specified independently, is the simplest for triple integrals. The most general volume of this type is a rectangular prism aligned with the coordinate axes, where each side is a rectangle parallel to one of the coordinate planes. The next two examples have a volumes which do not satisfy these criteria, with the result that the triple integrals become iterated integrals.

Example. Suppose that $f = xyz$, as in the preceding example, and V is the "wedge" shown in Fig. 3.2. We first determine the ranges of the integration variables. The wedge is bounded from above by the plane $y-z=0$, with all other bounding planes lying parallel to coordinate planes. Thus, the range of x is

$$0 \le x \le 1 \,. \tag{3.5}$$

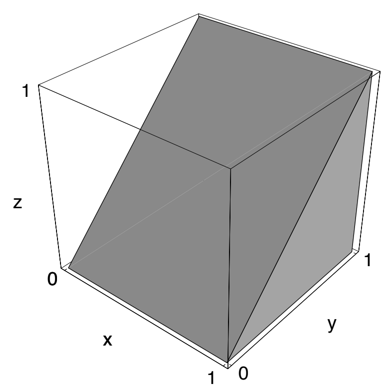

Figure 3.2: The volume for the triple integral in Eq. (3.9).

The triangular sides of the wedge are parallel to the plane $x = 0$, so the ranges of the y and z coordinates cannot be specified independently. Referring to Fig. 2.4, the ranges of these variables are

$$0 \le y \le 1 \,, \qquad 0 \le z \le y \,. \tag{3.6}$$

An alternative choice is (cf. Fig. 2.4)

$$z \le y \le 1 \,, \qquad 0 \le z \le 1 \,. \tag{3.7}$$

Using the ranges in Eq. (3.6), the triple integral is

$$\iiint_V f(x,y,z)dx\,dy\,dz = \int_0^1 x\,dx \int_0^1 y\,dy \int_0^y z\,dz \,. \tag{3.8}$$

As was the case for double integrals, this is called an **iterated integral** because the upper limit of the z-integral is y, which necessitates evaluating this integral *before* the y-integral. The x-integral can be carried out independently of the other two. Thus, carrying out the required integrations,

$$\underbrace{\int_0^1 x\,dx}_{\frac{1}{2}}\int_0^1 y\,dy\int_0^y z\,dz = \frac{1}{2}\int_0^1 y\,dy\int_0^y z\,dz$$

$$= \frac{1}{2}\int_0^1 y\,dy\underbrace{\left(\frac{1}{2}z^2\Big|_0^y\right)}_{\frac{1}{2}y^2} = \frac{1}{4}\underbrace{\int_0^1 y^3\,dy}_{\frac{1}{4}} = \frac{1}{16}\,. \tag{3.9}$$

The evaluation of this integral with the ranges in Eq. (3.7) is left as an exercise.

Example. Consider now the integration of $f = xyz$ over the volume in Fig. 3.3. This region is contained in the positive octant, bounded from below by the x-y plane and from above by the plane $x + y + z = 1$. The ranges of the integration variables are obtained by first observing that, in the x-y plane, where $z = 0$, the (x, y) coordinates within V are bounded by the line $x + y = 1$. Hence, the ranges of x and y may be chosen as

$$0 \le x \le 1\,, \quad 0 \le y \le 1 - x\,. \tag{3.10}$$

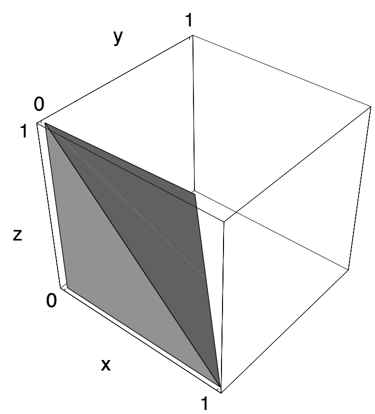

Figure 3.3: The for the triple integral in Eq. (3.4).

The lower bound for the range of z for all values of x and y is $z = 0$. The upper bound is obtained from the equation of the plane, solved for z: $z = 1 - x - y$. Hence,

$$0 \le z \le 1 - x - y\,, \tag{3.11}$$

so the integral to be evaluated is

$$\iiint_V f(x,y,z)dx\,dy\,dz = \int_0^1 x\,dx\int_0^{1-x} y\,dy\int_0^{1-x-y} z\,dz\,. \tag{3.12}$$

This is again an iterated integral in which the z-integration must be evaluated first, then the y-integration, and finally the x-integration. The integral over

z is evaluated as

$$\int_0^{1-x-y} z\,dz = \left(\frac{1}{2}z^2\bigg|_0^{1-x-y}\right) = \frac{1}{2}(1-x-y)^2 \tag{3.13}$$

By substituting this result into the y-integral and carrying out an integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^{1-x} y(1-x-y)^2\,dy \\
&= \underbrace{-\frac{1}{6}y(1-x-y)^3\bigg|_0^{1-x}}_{0} + \frac{1}{6}\int_0^{1-x}(1-x-y)^3\,dy \\
&= -\frac{1}{24}(1-x-y)^4\bigg|_0^{1-x} \\
&= \frac{1}{24}(1-x)^4\,.
\end{aligned} \tag{3.14}$$

Finally, substitution of this expression into the x-integral and again integrating by parts yields

$$\begin{aligned}
\frac{1}{24}\int_0^1 x(1-x)^4\,dx &= \underbrace{-\frac{1}{120}x(1-x)^5\bigg|_0^1}_{0} + \frac{1}{120}\int_0^1 (1-x)^5\,dx \\
&= -\frac{1}{720}(1-x)^5\bigg|_0^1 = \frac{1}{720}
\end{aligned} \tag{3.15}$$

as the value of the integral in Eq. (3.12). ▮

Cartesian coordinates are convenient for evaluating triple integrals within volumes bounded by planes. But there are many situations where other geometries are used, the most common of which are spheres and volumes contained within surfaces of revolution. In the next two sections, we will discuss two coordinate systems that considerably extend the capabilities of triple integrals.

3.2 Cylindrical Polar Coordinates

3.2.1 Definition of the Coordinate System

Cylindrical polar coordinates generalize circular polar coordinates (Sec. 2.2) to three dimensions by adding the "height" z to indicate the position of a point relative to the x-y plane [Fig. 3.4(a)]. The complete transformation from (x, y, z) to (r, ϕ, z) is

$$\boxed{x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = z\,,} \tag{3.16}$$

where

$$0 \le r < \infty, \qquad 0 \le \phi < 2\pi, \qquad -\infty < z < \infty\,. \tag{3.17}$$

This transformation is depicted in Fig. 3.4. The expressions for r and ϕ in terms of x and y are the same as those in Eq. (2.28).

3.2.2 The Integration Element

The integration element of this coordinate system can be obtained in two ways. The simplest way is to observe that the z-coordinate simply adds a

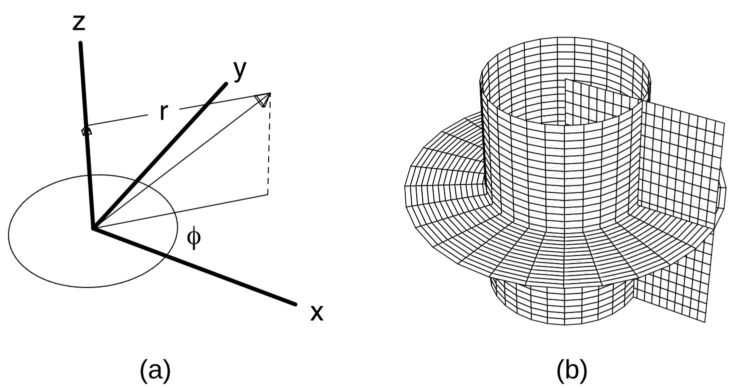

Figure 3.4: Two illustrations of circular polar coordinates. (a) The definitions of r, ϕ, and z. (b) The representation of any point as the intersection of the surface of constant r (the cylinder), constant ϕ (the vertical plane), and constant z (the horizontal plane).

"thickness" dz to the integration element in circular polar coordinates:

$$\boxed{dV = r\,dr\,d\phi\,dz\,.} \tag{3.18}$$

The other method, described in Problem Set 4, is based on writing any point (x, y, z) as a radius vector $\boldsymbol{r}$

$$\boldsymbol{r} = r\cos\phi\,\boldsymbol{i} + r\sin\phi\,\boldsymbol{j} + z\,\boldsymbol{k}\,, \tag{3.19}$$

and calculating the integration element from the vector product

$$dV = |d\boldsymbol{r}_r \cdot d\boldsymbol{r}_\phi \times d\boldsymbol{r}_z|\,, \tag{3.20}$$

where $d\boldsymbol{r}_r$, $d\boldsymbol{r}_\phi$, and $d\boldsymbol{r}_z$ are the differential changes of $\boldsymbol{r}$ with respect to r, ϕ, and z, respectively:

$$d\boldsymbol{r}_r = dr\cos\phi\,\boldsymbol{i} + dr\sin\phi\,\boldsymbol{j}\,, \tag{3.21}$$

$$d\boldsymbol{r}_\phi = -r\sin\phi\,d\phi\,\boldsymbol{i} + r\cos\phi\,d\phi\,\boldsymbol{j}\,, \tag{3.22}$$

$$d\boldsymbol{r}_z = dz\,\boldsymbol{k}\,. \tag{3.23}$$

3.2.3 Triple Integrals in Cylindrical Polar Coordinates

Example. Consider the sphere with unit radius in the upper half -space, as shown in Fig. 3.5. The equation of the surface is

$$x^2 + y^2 + z^2 = 1\,, \tag{3.24}$$

where $z \geq 0$. To calculate this volume as an integral in circular polar coordinates, we must first determine the ranges of the integration variables. The ranges of r and ϕ span the interior of the half-sphere:

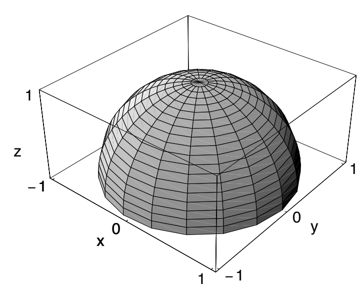

Figure 3.5: The unit upper half-sphere, $x^2 + y^2 + z^2 + 1$, for $z \geq 0$.

$$0 \leq r \leq 1\,, \qquad 0 \leq \phi < 2\pi \tag{3.25}$$

The upper bound for the range of z is obtained from the equation of the sphere, solved for z:

$$z^2 = 1 - x^2 - y^2 = 1 - r^2\,. \tag{3.26}$$

The half-sphere is bounded from below by the x-y plane, where $z = 0$. Hence, the range of z is

$$0 \le z \le \sqrt{1 - r^2}\,. \tag{3.27}$$

Thus, the volume integral of the half-sphere is given by

$$\begin{aligned} V &= \int_0^1 r\,dr \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\sqrt{1-r^2}} dz}_{\sqrt{1-r^2}} = 2\pi \int_0^1 r\sqrt{1-r^2}\,dr \\ &= 2\pi \left[-\frac{1}{3}(1-r^2)^{3/2} \Big|_0^1 \right] = 2\pi \times \frac{1}{3} = \frac{2\pi}{3}\,, \end{aligned} \tag{3.28}$$

which is one-half the volume of the unit sphere. ▮

Example. Consider the cone in Fig. 3.6. The surface is given by

$$x^2 + y^2 = (1 - z)^2\,, \tag{3.29}$$

for $0 \le z \le 1$. The ranges of r and ϕ are

$$0 \le r \le 1\,, \quad 0 \le \phi < 2\pi\,. \tag{3.30}$$

The range of z is calculated by following the steps in the preceding example. The cone is bounded from below by the x-y plane, where $z = 0$. The upper bound of x is determined by the surface of the cone which, in cylindrical coordinates, is $r^2 = (1 - z)^2$. Thus, the range of z is

$$0 \le z \le 1 - r\,. \tag{3.31}$$

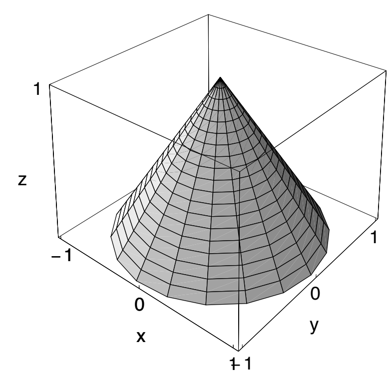

Figure 3.6: The surface determined by $x^2 + y^2 = (1 - z)^2$, for $0 \le z \le 1$.

The volume integral of the cone is

$$V = \int_0^1 r\,dr \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{1-r}}_{1-r} = 2\pi \int_0^1 (r - r^2)\,dr$$

$$= 2\pi \left(\frac{r^2}{2}\bigg|_0^1 - \frac{r^3}{3}\bigg|_0^1 \right) = 2\pi \times \frac{1}{6} = \frac{\pi}{3}\,. \tag{3.32}$$

∎

The preceding two examples showed how cylindrical polar coordinates are used to calculate the volumes of surfaces of revolution, i.e. surfaces that were obtained by rotating a curve about an axes, in those cases, the z-axis. We now consider a more substantial example by calculating the volume of another surface of revolution, the torus.

Example.* A torus, shown in Fig. 3.7, is a surface of revolution generated by rotating a circle of radius ρ whose center is a distance $R > \rho$ from the origin about an axis, usually taken as the z-axis. The calculation of the volume of a torus does not actually require an expression for the surface. The ranges of r, ϕ, and z can be determined by referring to Fig. 3.8. Consider first the left panel. Suppose we take the range of z as

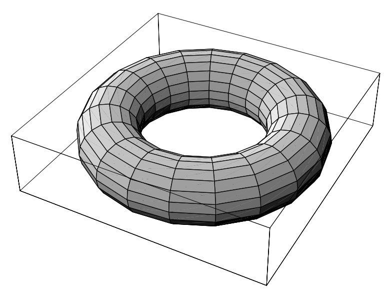

Figure 3.7: The surface of a torus.

$$-\rho \le z \le \rho\,. \tag{3.33}$$

The equation of the circle in the x-z plane is

$$(x - R)^2 + z^2 = \rho^2\,, \tag{3.34}$$

so the range of r is obtained by solving this equation for x and referring to Fig. 3.8(b):

$$R - \sqrt{\rho^2 - z^2} \le r \le R + \sqrt{\rho^2 - z^2}\,. \tag{3.35}$$

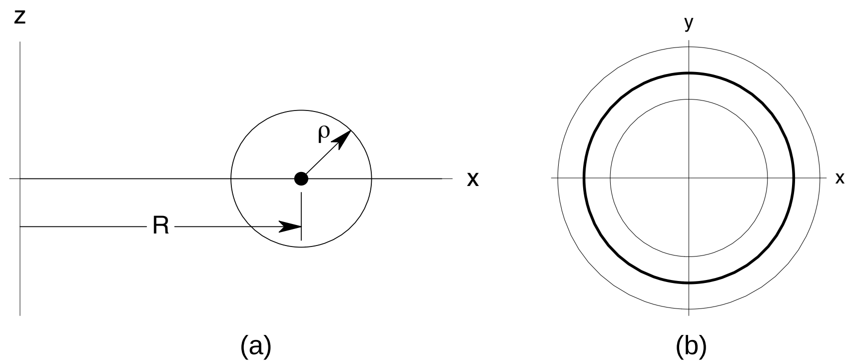

Figure 3.8: (a) The circle in the x-z plane that is rotated about the z-axis. (b) The section of the torus in the x-y plane. The emboldened line is the path traced out by the center of the circle.

Figure 3.8(b) indicates that the range of ϕ is

$$0 \le \phi \le 2\pi\,. \tag{3.36}$$

The volume integral of the torus is thus given by

$$V = \int_0^{2\pi} d\phi \int_{-\rho}^{\rho} dz \int_{R-\sqrt{\rho^2-z^2}}^{R+\sqrt{\rho^2-z^2}} r\,dr\,. \tag{3.37}$$

The radial integral is evaluated as

$$\begin{aligned} \int_{R-\sqrt{\rho^2-z^2}}^{R+\sqrt{\rho^2-z^2}} r\,dr &= \frac{r^2}{2}\bigg|_{R-\sqrt{\rho^2-z^2}}^{R+\sqrt{\rho^2-z^2}} \\ &= \frac{1}{2}\left[\left(R+\sqrt{\rho^2-z^2}\right)^2 - \left(R-\sqrt{\rho^2-z^2}\right)^2\right] \\ &= 2R\sqrt{\rho^2-z^2}\,. \end{aligned} \tag{3.38}$$

The integral over the azimuthal angle in Eq. (3.37) is 2π, so the volume integral reduces to

$$V = 4\pi R \int_{-\rho}^{\rho} \sqrt{\rho^2-z^2}\,dz\,. \tag{3.39}$$

This integral can be evaluated by the trigonometric substitution $z = \rho\sin\theta$, where $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$. Carrying out the required changes to the integrand, the integration element, and the limits of integration yields

$$V = 4\pi R\int_{-\rho}^{\rho}\sqrt{\rho^2 - z^2}\,dz = 4\pi R\rho^2 \underbrace{\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos^2\theta\,d\theta}_{\frac{1}{2}\pi} = 2\pi^2 R\rho^2\,. \tag{3.40}$$

By writing this result as

$$V = (2\pi R)\times(\pi\rho^2)\,, \tag{3.41}$$

the volume of a torus can be interpreted as the product of the area of the circle that is rotated about the z-axis to form the torus ($\pi\rho^2$) and the length of the path taken by the center of the circle ($2\pi R$). This is a special case of **Pappus' Theorem**[1]: let R be a planar region that lies entirely on one side of an axis (usually the z-axis) in the plane. If R is rotated about this axis, the volume of the resulting solid is the product of the area A of R and the distance travelled by its centroid. ∎

3.3 Spherical Polar Coordinates

One of the most important coordinate systems in physics is spherical polar coordinates. These are appropriate whenever there is a spherical boundary or a section of such a boundary. Spherical polar coordinates are especially important for the quantum mechanical theory of atoms, which is based on spherical symmetry.

3.3.1 Definition of the Coordinate System

In spherical polar coordinates, a point (x, y, z) is expressed in terms of the **radius** r, which measures the distance of the point from the origin, the **azimuthal angle** ϕ, which measures the orientation of the radius vector with respect to

[1]Pappus of Alexandria, who lived in the 4th century, is considered to be the last of the great Greek geometers.

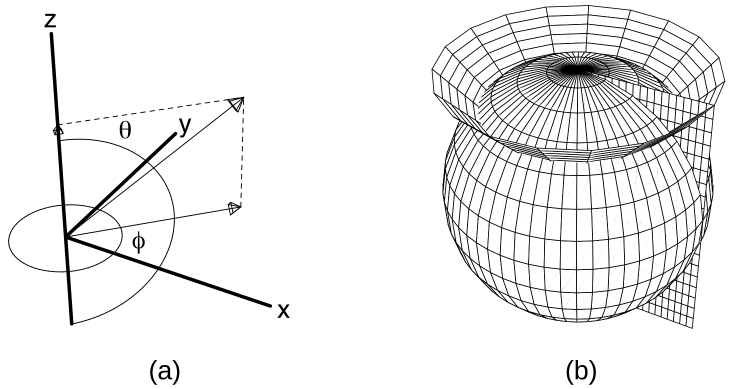

Figure 3.9: Two depictions of spherical polar coordinates. (a) The definitions and ranges of r, ϕ, and θ. (b) The representation of any point as the intersection of the surface of constant r (the sphere), constant ϕ (the plane), and constant θ (the cone).

the positive x-axis, with positive ϕ taken in the counterclockwise direction, and the **polar angle** θ, which measures the orientation of the radius vector with respect to the z-axis. These definitions and conventions are depicted in Fig. 3.9.

The transformations between the coordinates (x, y, z) and (r, ϕ, θ) is determined from the trigonometric construction in Fig. 3.10. The projection of the radial vector onto the x-y plane has length $r \sin\theta$. The x and y coordinates are obtained by projecting this quantity onto the x and y axes:

$$x = r \sin\theta \cos\phi \,, \tag{3.42}$$

$$y = r \sin\theta \sin\phi \,. \tag{3.43}$$

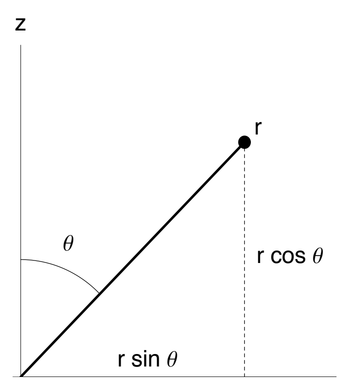

Figure 3.10: The orientation of a radial vector with respect to the z-axis.

The projection of the radius onto the z-axis is

$$z = r \cos\theta \,. \tag{3.44}$$

These are the transformations that relate Cartesian coordinates to spherical polar coordinates.

The ranges of the radial and azimuthal variables are determined by referring to Fig. 3.9(a). As in circular polar coordinates (Sec. 2.2)

$$0 \le r < \infty\,, \qquad 0 \le \phi \le 2\pi\,. \tag{3.45}$$

The range of θ is determined by requiring that the transformation between Cartesian and spherical polar coordinates is *single-valued*, i.e. that one and only one set of spherical polar coordinates (r, ϕ, θ) corresponds to a particular set of Cartesian coordinates (x, y, z). This necessitates restricting the range of θ to

$$0 \le \theta \le \pi\,. \tag{3.46}$$

To understand this, consider a point $\boldsymbol{r}$

$$\boldsymbol{r} = r\cos\phi\sin\theta\,\boldsymbol{i} + r\sin\phi\sin\theta\,\boldsymbol{j} + r\cos\theta\,\boldsymbol{k}\,. \tag{3.47}$$

Suppose that we transform this point by rotating the azimuthal angle by π: $\phi \to \phi + \pi$. According to the Eq. (3.46), this is *not* an allowed rotation. The coordinates of the transformed point $\boldsymbol{r}'$ are obtained by applying standard trigonometric identities:

$$\boldsymbol{r}' = -r\cos\phi\sin\theta\,\boldsymbol{i} - r\sin\phi\sin\theta\,\boldsymbol{j} + r\cos\theta\,\boldsymbol{k}\,. \tag{3.48}$$

Now suppose that we rotate the polar angle by π: $\theta \to \theta + \pi$. The coordinates of the transformed point $\boldsymbol{r}''$ are again determined by applying standard trigonometric identities:

$$\boldsymbol{r}'' = -r\cos\phi\sin\theta\,\boldsymbol{i} - r\sin\phi\sin\theta\,\boldsymbol{j} + r\cos\theta\,\boldsymbol{k}\,. \tag{3.49}$$

By comparing these coordinates with those in Eq. (3.48), we conclude that $\boldsymbol{r}'' = \boldsymbol{r}'$, i.e. that there are two ways of labelling the same point. To avoid this unacceptable result, the range of θ is restricted to the range in Eq. (3.46).

3.3.2 The Integration Element

The integration element in spherical polar coordinates is most easily obtained with the procedure in Problem Set 4. The radius vector associated with a point (x, y, z) is written as

$$\boldsymbol{r} = r\cos\phi\sin\theta\,\boldsymbol{i} + r\sin\phi\sin\theta\,\boldsymbol{j} + r\cos\theta\,\boldsymbol{k}\,, \tag{3.50}$$

and calculating the integration element from the vector product

$$dV = |d\boldsymbol{r}_r \cdot d\boldsymbol{r}_\phi \times d\boldsymbol{r}_\theta| \,, \tag{3.51}$$

where $d\boldsymbol{r}_r$, $d\boldsymbol{r}_\phi$, and $d\boldsymbol{r}_\theta$ are the differential changes of $\boldsymbol{r}$ with respect to r, ϕ, and z, respectively:

$$d\boldsymbol{r}_r = dr\cos\phi\sin\theta\,\boldsymbol{i} + dr\sin\phi\sin\theta\,\boldsymbol{j} + dr\cos\theta\,\boldsymbol{k}\,, \tag{3.52}$$

$$d\boldsymbol{r}_\phi = -r\sin\phi\sin\theta\,d\phi\,\boldsymbol{i} + r\cos\phi\sin\theta\,d\phi\,\boldsymbol{j}\,, \tag{3.53}$$

$$d\boldsymbol{r}_\theta = r\cos\phi\cos\theta\,d\theta\,\boldsymbol{i} + r\sin\phi\cos\theta\,d\theta\,\boldsymbol{j} + -r\sin\theta\,d\theta\,\boldsymbol{k}\,. \tag{3.54}$$

These vectors are mutually orthogonal so the integration element is obtained from the product of their magnitudes:

$$\boxed{dV = r^2\sin\theta\,dr\,d\phi\,d\theta\,.} \tag{3.55}$$

3.3.3 Triple Integrals in Spherical Polar Coordinates

Integrals of a function $F(x,y,z)$ over a volume V are written in spherical polar coordinates as

$$\begin{aligned}
&\iiint_V F(x,y,z)\,dx\,dy\,dz \\
&= \iiint_{V'} F\left[x(r,\phi,\theta),y(r,\phi,\theta),z(r,\phi,\theta)\right]\,r^2\sin\theta\,dr\,d\phi\,d\theta \\
&\equiv \iiint_{V'} f(r,\phi,\theta)\,r^2\sin\theta\,dr\,d\phi\,d\theta\,,
\end{aligned} \tag{3.56}$$

where V' is the volume V expressed in spherical polar coordinates. There are two important special cases of this integral. If f has no ϕ-dependence, $f = f(r,\theta)$, then f is said to have **azimuthal symmetry**. According to the transformations in Eqs. (3.43) and (3.44) and Fig. 3.8, this corresponds to rotational symmetry about the z-axis. Surfaces of revolution have this type of symmetry. A physical situation with this type of symmetry is discussed in Problem Set 4. The integral over ϕ can be evaluated immediately and the general expression in Eq. (3.56) becomes

$$2\pi \iint f(r,\theta)\, r^2 sin\theta\, dr\, d\theta\,. \tag{3.57}$$

In the second case, where f has neither ϕ- nor θ-dependence, f is said to be **isotropic**. This corresponds to "spherical" symmetry in that f depends only on the radius r and not on any angular orientation. The integrals over ϕ and θ can be evaluated immediately and the general integral in Eq. (3.56) reduces to

$$4\pi \int f(r) r^2\, dr\,. \tag{3.58}$$

This integral is seen to correspond to the integration over radial shells.

Example. We consider first the calculation of the volume of a sphere of radius R. Referring to Eq. (3.56), this corresponds to the case $f = 1$. The ranges of the integration variables are obtained directly from Fig. 3.9(a):

$$0 \le r \le R\,, \qquad 0 \le \phi < 2\pi\,, \qquad 0 \le \theta \le \pi\,, \tag{3.59}$$

so the volume integral is

$$V = \underbrace{\int_0^R r^2\, dr}_{\frac{1}{3}R} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi} \sin\theta\, d\theta}_{2} = \frac{4}{3}\pi R^3\,. \tag{3.60}$$

The generalization of this procedure to sections of a sphere between given azimuthal and polar angles and to spherical shells with given inner and out radii is straightforward. ▮

Example. Consider the integral of $f = e^{-\alpha r}$ over all space. This is an example of a function with spherical symmetry that occurs frequently in quantum mechanics. The ranges of the integration variables are

$$0 \le r < \infty\,, \qquad 0 \le \phi < 2\pi\,, \qquad 0 \le \theta \le \pi\,, \tag{3.61}$$

so the integral of f becomes

$$\int_0^{\infty} r^2 e^{-\alpha r}\, dr \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi} \sin\theta\, d\theta}_{2} = 4\pi \int_0^{\infty} r^2 e^{-\alpha r}\, dr\,. \tag{3.62}$$

The radial integral is evaluated by performing successive integrations by parts:

$$\begin{aligned}
\int_0^\infty \underbrace{r^2}_{u}\,\underbrace{e^{-\alpha r}\,dr}_{dv} &= \underbrace{-\frac{1}{\alpha}r^2 e^{-\alpha r}\bigg|_0^\infty}_{0} + \frac{2}{\alpha} r e^{-\alpha r}\,dr \\
&= \frac{2}{\alpha}\bigg(\underbrace{-\frac{1}{\alpha} r e^{-\alpha r}\bigg|_0^\infty}_{0} + \frac{1}{\alpha}\int_0^\infty e^{-\alpha r}\,dr\bigg) \\
&= \frac{2}{\alpha^2}\left(-\frac{1}{\alpha}e^{-\alpha r}\bigg|_0^\infty\right) \\
&= \frac{2}{\alpha^3}\,.
\end{aligned} \tag{3.63}$$

Notice that, in arriving at this result, we have twice used the fact that

$$\lim_{x\to\infty} x^n e^{-\alpha x} = 0 \tag{3.64}$$

for any n.

3.4 Surface Integrals

A particular case of integrals in three dimensions involves integrals over surfaces. A common type of surface integral is where one of the three variables is held constant. Consider the surface of a sphere of radius R. According to Eq. (3.50) the radius $\boldsymbol{r}$ vector at any point on the sphere is

$$\boldsymbol{r} = R\cos\phi\sin\theta\,\boldsymbol{i} + R\sin\phi\sin\theta\,\boldsymbol{j} + R\cos\theta\,\boldsymbol{k}\,, \tag{3.65}$$

The element of area integration is obtained by calculating the differential of this vector for changes in turn of $d\phi$ and $d\theta$:

$$d\boldsymbol{r}_\phi = -R\sin\phi\sin\theta\,d\phi\,\boldsymbol{i} + R\cos\phi\sin\theta\,d\phi\,\boldsymbol{j}\,, \tag{3.66}$$

$$d\boldsymbol{r}_\theta = R\cos\phi\cos\theta\,d\theta\,\boldsymbol{i} + R\sin\phi\cos\theta\,d\theta\,\boldsymbol{j} + -R\sin\theta\,d\theta\boldsymbol{k}\,. \tag{3.67}$$

These vectors are orthogonal,

$$d\boldsymbol{r}_\phi \cdot d\boldsymbol{r}_\theta = 0\,, \tag{3.68}$$

so the differential area dA corresponding to these differential changes is obtained from the product of the magnitudes of $d\boldsymbol{r}_\phi$ and $d\boldsymbol{r}_\phi$:

$$dA = |d\boldsymbol{r}_\phi||d\boldsymbol{r}_\theta| = R\sin\theta\, d\phi \times R\, d\theta = R^2 \sin\theta\, d\phi\, d\theta\,. \tag{3.69}$$

Example. The surface area of a sphere of radius R is represented as

$$R^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta = R^2 \times 2\pi \times 2 = 4\pi R^2\,. \tag{3.70}$$

The corresponding expression of the surface area subtended by azimuthal angles Φ_1 and Φ_2 and polar angles Θ_1 and Θ_2 is

$$R^2 \int_{\Phi_1}^{\Phi_2} d\phi \int_{\Theta_1}^{\Theta_2} \sin\theta\, d\theta = R^2(\Phi_2 - \Phi_1)(\cos\Theta_1 - \cos\Theta_2)\,. \tag{3.71}$$

The other type of surface integral we will encounter involve a cylinder of radius R. The radius vector is, from Eq. (3.19), given by

$$\boldsymbol{r} = R\cos\phi\, \boldsymbol{i} + R\sin\phi\, \boldsymbol{j} + z\, \boldsymbol{k}\,, \tag{3.72}$$

The differential $d\boldsymbol{r}$ corresponding to differential changes of $d\phi$ and dz are

$$d\boldsymbol{r}_\phi = -R\sin\phi\, d\phi\, \boldsymbol{i} + R\cos\phi\, d\phi \boldsymbol{j}\,, \tag{3.73}$$

$$d\boldsymbol{r}_z = dz\, \boldsymbol{k}\,. \tag{3.74}$$

These vectors are manifestly orthogonal, $d\boldsymbol{r}_\phi \cdot d\boldsymbol{r}_z = 0$, so the differential area dA corresponding to these differential changes is obtained from the product of the magnitudes of $d\boldsymbol{r}_\phi$ and $d\boldsymbol{r}_z$:

$$dA = R\, d\phi\, dz\,. \tag{3.75}$$

Example. The surface of a cylinder of radius R and height H is calculated as

$$R\int_0^{2\pi} d\phi \int_0^H dz = 2\pi RH\,. \tag{3.76}$$

The surface area of cylinder between heights H_1 and H_2 and azimuthal angles Φ_1 and Φ_2 is similarly calculated as

$$R\int_{\Phi_1}^{\Phi_2} d\phi \int_{H_1}^{H_2} dz = R(\Phi_2 - \Phi_1)(H_2 - H_1)\,. \tag{3.77}$$

3.5 Summary

The triple integral of a function $f(x, y, z)$, viewed as a density of some physical quantity, is the amount of that quantity within a volume in three-dimensional space. There is considerably more freedom to specify other coordinate systems than in two dimensions and many applications in physics rely on such transformations to enable calculations to be carried out. From Cartesian coordinates, we transformed triple integrals into cylindrical polar coordinates, which are the natural generalizations of circular polar coordinates to three dimensions, and are appropriate to situations where there is azimuthal symmetry, and spherical polar coordinates, for situations that involve spherical symmetry. The Jacobians obtained in each case reflect the position dependence of the magnitude of the differential volume elements.