Chapter 2

Double Integrals

Quantities such as mass and charge density are often defined for systems within continuous regions of two- or three-dimensional space. Three-dimensional systems are the typical case, but two-dimensional systems are of interest either as limiting cases of three dimensions (such as thin plates), or as inherently low-dimensional systems (such as electrons within nano-structured materials that are a few atoms wide). Determining the amount of a physical quantity within a given region requires performing integrals over that region. This leads to the notion of double and triple integrals. These integrals are natural extensions of Riemann integrals in one dimension that were discussed in Sec. 1.2. In this chapter, we discuss the evaluation of double integrals, initially in Cartesian coordinates, and then in circular polar coordinates, which find frequent application when dealing with circular boundaries.

2.1 Integrals in Cartesian Coordinates

Consider a two-dimensional system within which a function f is defined at every point (x, y). This function can represent a physical quantity such as a mass or charge density, the local temperature or, more abstractly, a probability density, which occurs in quantum mechanics and in the physics of random processes. Calculating the amount of this quantity within a specified region A in the x-y plane necessitates performing an integral over the ranges of x and y that span the interior of A. Because this operation involves the

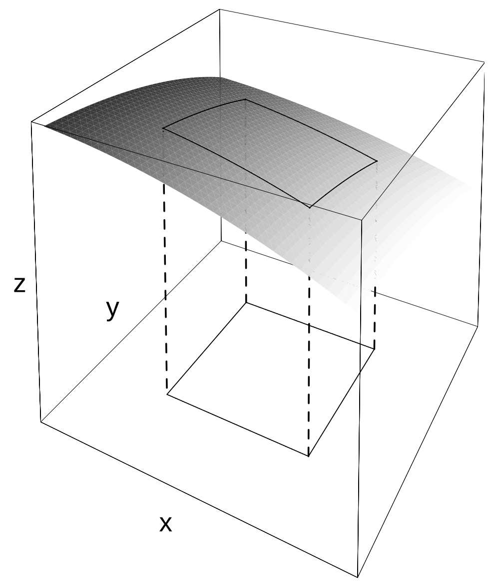

Figure 2.1: Geometric representation of a double integral of a function $f(x, y)$, represented as the surface $z = f(x, y)$ (shown shaded). The integral is bounded by integration range A in the x-y plane, the corresponding region mapped onto the surface, and the vertical extensions of the boundaries of A to f. This interpretation is analogous to that in Fig. 1.3 for one-dimensional integrals.

integration over two variables, it is represented as a **double integral**:

$$\boxed{\iint_A f(x, y)\, dx\, dy\,.} \tag{2.1}$$

Double integrals have a geometrical interpretation that is analogous to that of one-dimensional integrals. As shown in Fig. 2.1, the function $f(x, y)$ can be represented as the surface $z = f(x, y)$ in three-dimensional space. The region A in the x-y plane is mapped to the region $f(A)$. The integral in Eq. (2.1) therefore corresponds to the volume bounded by the x-y plane, the surface $f(A)$, and the boundaries of A extended vertically to f.

There are several important points to note about double integrals:

1. Once the area A is specified, the integral has a unique value.

2. The integrations over x and y can be carried out in any order.

3. For the special case $f(x, y) = 1$, the region of integration is a cylinder of unit height with base area A. The value of the integral is, therefore, the area of A.

Example. Suppose $f = x^2 y$ and A is the rectangular region shown in Fig. 2.2. The evaluation of the double integral

$$\iint_A x^2 y \, dx \, dy \tag{2.2}$$

proceeds by first determining the ranges of x and y for the coordinates of every point within A. Since the boundaries of A are parallel to the x- and y-axes, these ranges are readily determined as

$$1 \le x \le 3 \,, \qquad 1 \le y \le 2 \,. \tag{2.3}$$

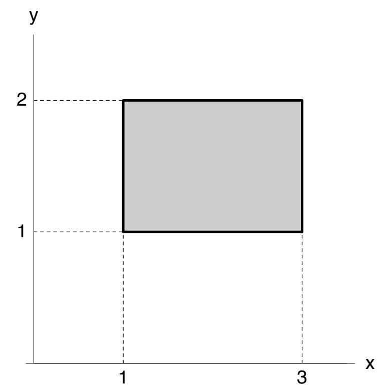

Figure 2.2: The integration region A, shown shaded, for the double integral in Eq. (2.2).

The double integral to be evaluated is

$$\int_1^3 dx \int_1^2 dy \, x^2 y = \int_1^3 x^2 \, dx \int_1^2 y \, dy \,. \tag{2.4}$$

The original integral has thereby been reduced to two one-dimensional integrals. This is a general feature of multiple integrals: their evaluation always reduces to a sequence of one-dimensional integrals. The final step is the evaluation of the integrals in Eq. (2.4):

$$\int_1^3 x^2 \, dx = \frac{1}{3} x^3 \Big|_1^3 = 9 - \frac{1}{3} = \frac{26}{3} \,, \tag{2.5}$$

$$\int_1^2 y \, dy = \frac{1}{2} y^2 \Big|_1^2 = 2 - \frac{1}{2} = \frac{3}{2} \,, \tag{2.6}$$

from which we obtain

$$\iint_A x^2 y \, dx \, dy = \frac{26}{3} \times \frac{3}{2} = 13 \,. \tag{2.7}$$

∎

Double integrals over regions bounded by lines that are parallel to the coordinate axes are especially straightforward to evaluate because the ranges of x and y are independent of one another. But, as the following example shows, this is not always the case.

Example. Suppose that, in the double integral in Eq. (2.2), the integration region A is the triangle shown in Fig. 2.3. Identifying the ranges of x and y for this region requires a different procedure from that in Fig. 2.2. There are two ways that this integral can be done: by carrying out the y-integration first followed by the x-integration, and by carrying out the x-integration first followed by the y-integration.

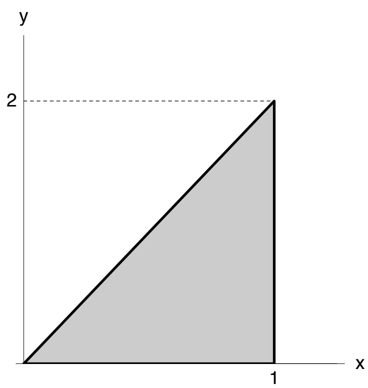

Figure 2.3: The integration region A, shown shaded, for the double integral in Eq. (2.2).

Method I. If x is allowed to range over the interval $0 \le x \le 1$ then, as shown in Fig. 2.4(a), the values of y corresponding to a particular value of x must lie in the range $0 \le y \le 2x$ because A is bounded from below by the x-axis and from above by the line $y = 2x$. Thus, the double integral over A is written as

$$\int_0^1 dx \int_0^{2x} dy\, x^2 y = \int_0^1 x^2\, dx \int_0^{2x} y\, dy = \int_0^1 \left(x^2 \int_0^{2x} y\, dy \right) dx\,. \tag{2.8}$$

Notice that, because the upper limit of the y-integration is a function of x, this integral must be performed *before* the integral over x. Such a multiple integral is called an **iterated integral**. In effect, the double integral over A has been represented as an integral over each vertical strip that runs parallel to the y-axis within A, followed by an integral over all of these strips. Accordingly, the integration over y yields

$$\int_0^{2x} y\, dy = \frac{1}{2}\, y^2 \Big|_0^{2x} = 2x^2\,, \tag{2.9}$$

The integral over x can now be carried out, and we obtain

$$2\int_0^1 x^4\, dx = \frac{2}{5}\, x^5 \Big|_0^1 = \frac{2}{5}\,. \tag{2.10}$$

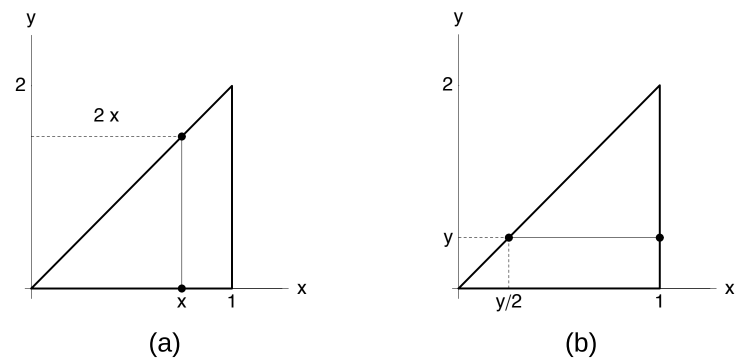

Figure 2.4: Two ways of setting up the ranges of integration for the region shown in Fig. 2.3. (a) The allowed values of y for a given value of x in the range $0 \le x \le 1$. (b) The allowed values of x for a given value of y in the range $0 \le y \le 1$.

Method II. This integral can also be evaluated by performing the integration over x first. Referring to Fig. 2.4(b), the range of y is $0 \le y \le 2$. For a given value of y, the corresponding values of x within A are bounded from the left by $y = 2x$ and from the right by $x = 1$. Thus, the corresponding range of x is $\frac{1}{2}y \le x \le 1$. The double integral is now written as

$$\int_0^2 dy \int_{\frac{1}{2}y}^1 dx\, x^2 y = \int_0^2 y\, dy \int_{\frac{1}{2}y}^1 x^2\, dx = \int_0^2 \left(y \int_{\frac{1}{2}y}^1 x^2\, dx \right) dy\,. \tag{2.11}$$

The integration over x must now be performed first, with the result

$$\int_{\frac{1}{2}y}^1 x^2\, dx = \frac{1}{3}\, x^3 \Big|_{\frac{1}{2}y}^1 = \frac{1}{3} - \frac{1}{24} y^3\,. \tag{2.12}$$

The integration over y then yields

$$\begin{aligned} \frac{1}{3}\int_0^2 y\, dy - \frac{1}{24}\int_0^2 y^4\, dy &= \frac{1}{6}\, y^2\Big|_0^2 - \frac{1}{120} y^5\Big|_0^2 \\ &= \frac{2}{3} - \frac{32}{120} \\ &= \frac{2}{5}\,, \end{aligned} \tag{2.13}$$

which agrees with the result in Eq. (2.10). ∎

The approach described in the preceding examples can be applied to any region bounded by straight line segments. In some cases, the same methods can be used for regions with circular boundaries. The following example shows how to calculate the area of a semi-circular region.

Example. Consider the integral

$$\iint_A dx\,dy\,, \qquad (2.14)$$

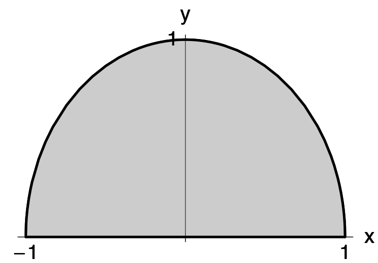

Figure 2.5: The semi-circular region A, shown shaded, for the double integral in Eq. (2.14).

where the area A, shown in Fig. 2.5, is bounded from below by the x-axis, and from above by the boundary of the circle $x^2 + y^2 = 1$. Because the integrand is unity, the value of this integral is equal to the area of A. We will evaluate this integral by performing the integral over y first. The range of x is $-1 \le x \le 1$. For a given value of x, the values of y within A are bounded from below by the x-axis and from above by the circular boundary. Thus, the range of y is $0 \le y \le \sqrt{1-x^2}$. The double integral is therefore written as

$$\int_{-1}^{1} dx \int_0^{\sqrt{1-x^2}} dy\,. \qquad (2.15)$$

The integral over y is straightforward to carry out and we obtain

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx\,. \qquad (2.16)$$

This integral can be evaluated by trigonometric substitution. We set $x = \sin\phi$. Then,

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\phi} = \cos\phi\,, \qquad (2.17)$$

$$dx = \cos\phi\,d\phi\,, \qquad (2.18)$$

and the limits of integration are transformed as

$$x = -1 \longrightarrow \phi = -\tfrac{1}{2}\pi\,, \qquad x = 1 \longrightarrow \phi = \tfrac{1}{2}\pi\,. \qquad (2.19)$$

The transformed integral is

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^2\phi\, d\phi = \tfrac{1}{2}\pi\,, \tag{2.20}$$

which is the area of the semi-circular region. ▮

With the examples in this section as background, we can summarize the evaluation of double integrals over any region A in the x-y plane by the two approaches illustrated in Fig. 2.5. In Fig. 2.6(a), the range of x is $x_A \le x \le x_B$ and the corresponding range of y at a particular value of $x$is $u_1(x) \le y \le u_2(x)$, and the double integral is written as

$$\boxed{\iint_A f(x,y)\, dx\, dy = \int_{x_A}^{x_B} dx \int_{u_1(x)}^{u_2(x)} dy\, f(x,y)\,.} \tag{2.21}$$

In Fig. 2.6(b), the range of y is $y_A \le y \le y_B$ and the corresponding range of x for a particular value of y is $v_1(y) \le x \le v_2(y)$, and the double integral is

$$\boxed{\iint_A f(x,y)\, dx\, dy = \int_{y_A}^{y_B} dy \int_{v_1(y)}^{v_2(y)} dx\, f(x,y)\,.} \tag{2.22}$$

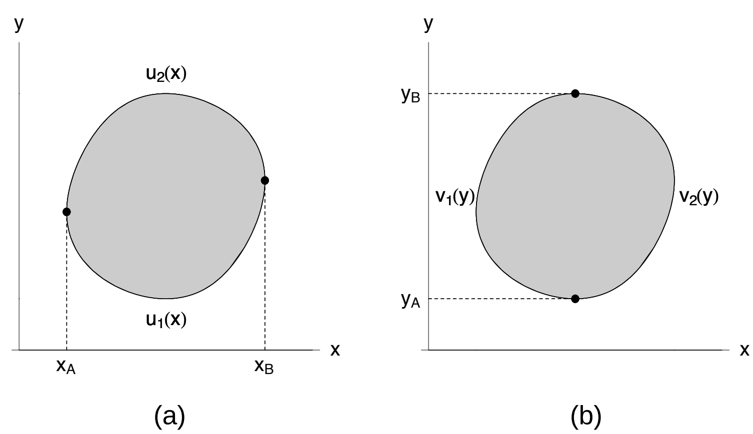

Figure 2.6: The two methods of evaluating a double integral over a region in the x-y plane, shown shaded.

Although these expressions indicate the order in which the integrals over x and y are to be carried out, the actual evaluation of these integrals may prove problematic for certain types of boundaries and integrands. For some common cases there are special methods available. We consider an example.

Example. Consider the integral

$$\iint_A e^{-x^2-y^2}\, dx\, dy\,, \tag{2.23}$$

where the region A is shown in Fig. 2.5. In integrals of this type arise in quantum mechanics and in the physics of random processes. We set up the integral using the same steps that lead to Eq. (2.15):

$$\int_{-1}^{1} dx \int_0^{\sqrt{1-y^2}} dy\, e^{-x^2-y^2} = \int_{-1}^{1} e^{-x^2}\, dx \int_0^{\sqrt{1-y^2}} e^{-y^2}\, dy\,. \tag{2.24}$$

We arrive at an impasse because there is no explicit expression for the primitive function of e^{-y^2}. The problem is not the boundary, but the integrand. In fact, the semi-circular boundary provides the basis for an alternative way of writing this integral that enables it to be evaluated in a straightforward manner. This involves the transformation to a new coordinate system and will be discussed in the next section.

2.2 Circular Polar Coordinates

2.2.1 Definitions of Coordinate Systems

The ability to transform integrals and derivatives between different coordinate systems is one of the cornerstones of modern physics. Cartesian coordinates provide a simple way of labelling any point in the x-y plane. Lines parallel to the coordinate axes are drawn from the point and their intersection points with the x and y axes define the coordinates of that point, as shown in Fig. 2.7(a). The ranges of x and y are

$$-\infty < x < \infty\,, \qquad -\infty < y < \infty\,, \tag{2.25}$$

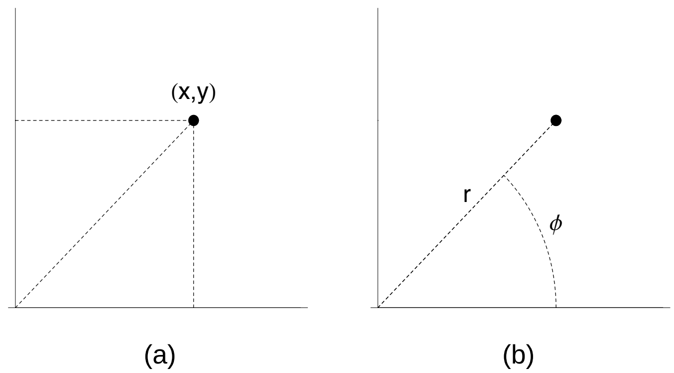

Figure 2.7: Two ways of labelling the same point in the plane: (a) (x, y) in a Cartesian coordinate system and (b) (r, ϕ) in a circular polar coordinate system.

include all points in the x-y plane. This coordinate system is conceptually simple and has natural extensions to higher dimensions.

But there are other ways of labelling points that may be more suitable in particular circumstances. The basic idea of **circular polar coordinates** is to specify any point (x, y) in terms of two new variables: (i) a **radius** r that specifies the distance of the point from the origin, and (ii) an angular variable ϕ, called the **azimuthal angle**, that specifies the angle of the radial vector with respect to some axis, by convention taken to be the positive x-axis, with the angle increasing from zero in the counterclockwise direction. These quantities are shown in Fig. 2.7(b). The variable r is an inherently non-negative quantity so its range is

$$0 \le r < \infty\,. \tag{2.26}$$

The azimuthal angle must account for all orientations with respect to the positive x-axis while maintaining a unique labelling for all points, so its range is

$$0 \le \phi < 2\pi\,. \tag{2.27}$$

The relationship between the two coordinate systems can be determined from a standard trigonometric analysis:

$$x = r\cos\phi\,, \qquad y = r\sin\phi\,, \tag{2.28}$$

or, alternatively,

$$r = \sqrt{x^2 + y^2}\,, \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right)\,. \tag{2.29}$$

The restriction of the range of ϕ in Eq. (2.27) can now be understood as restricting the trigonometric functions in Eq. (2.28) to a single period.

The differences between the Cartesian and circular polar coordinates are best appreciated pictorially by plotting the coordinate curves, i.e. the curves where one of the coordinates is held constant. The coordinate curves of the Cartesian coordinate system are straight lines parallel to the x and y axes, as shown in Fig. 2.8(a). For the circular polar coordinate system, the curves of constant r are, from Eq. (2.29) concentric circles centered at the origin. The lines of constant azimuthal angle are straight lines through the origin that make an angle ϕ with respect to the positive x-axis. These are shown in Fig. 2.8(b).

2.2.2 Transformation of the Integration Element

The transformation from (x, y) to (r, ϕ) necessitates changes to the element of integration in double integrals. The basic calculation involves differential changes of the variables and calculating the area enclosed as the result of

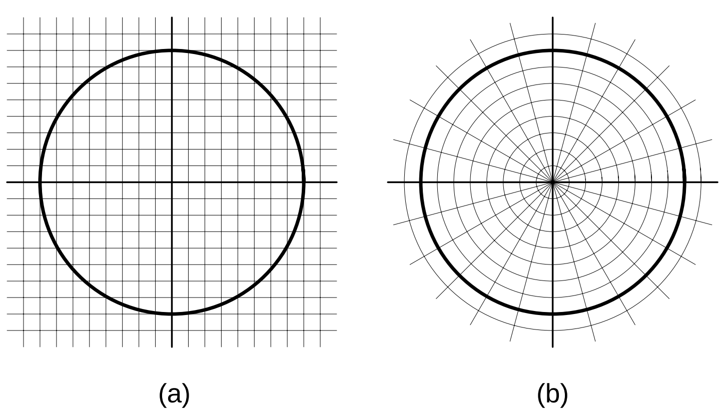

Figure 2.8: The coordinate curves of the (a) Cartesian and (b) circular polar coordinate system. Also shown is a circle to indicate how the polar coordinates provide a more natural description of such boundaries than the Cartesian coordinates.

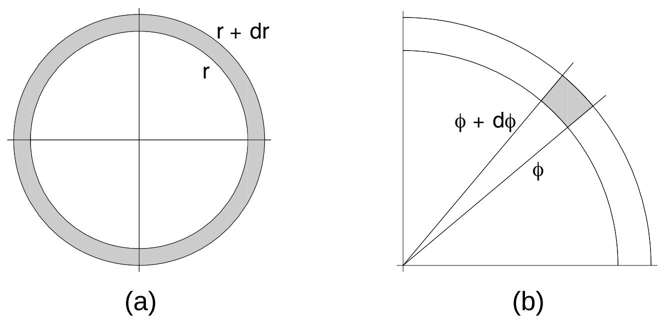

Figure 2.9: The steps used to calculate the integration element in circular polar coordinates. The area between the circles of radii r and $r + dr$ is shown in the left panel. The fraction of this area contained between the azimuthal angles ϕ and $\phi + d\phi$ is shown in the right panel.

these changes. For Cartesian coordinates, at any point (x, y), changing x to $x + dx$ and y to $y + dy$ encloses an area $dx\,dy$ everywhere in the x-y plane. We now consider the corresponding calculation in circular polar coordinates, where we calculate the area enclosed by the differential changes $r \to r + dr$ and $\phi \to \phi + d\phi$. The steps are shown in Fig. 2.9. The area between the circles of radii r and $r + dr$, depicted in the left panel of Fig. 2.9, which is calculated either as $\pi(r+dr)^2 - \pi r^2$, or directly as the differential of the area of a circle of radius r:

$$d(\pi r^2) = 2\pi r\,dr\,. \tag{2.30}$$

The fraction of this area contained between the azimuths ϕ and $\phi + d\phi$ is $d\phi/2\pi$, i.e. the fraction of 2π that $d\phi$ represents. Hence, the area dA enclosed by the infinitesimal variations of r and ϕ is

$$dA = 2\pi r\,dr \times \frac{d\phi}{2\pi} = r\,dr\,d\phi\,. \tag{2.31}$$

Note the factor of r multiplying $dr\,d\phi$. This results from the fact that, for a fixed angle $d\phi$, the arc length between ϕ and $\phi + d\phi$ is $r\,d\phi$.

An alternative way of deriving the integration element is to first construct the vector $\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$ which, in terms of the variables in Eq. (2.28) is

$$\boldsymbol{r} = r\cos\phi\,\boldsymbol{i} + r\sin\phi\,\boldsymbol{j}\,. \tag{2.32}$$

We now calculate the vectors $d\boldsymbol{r}_r$ and $d\boldsymbol{r}_\phi$ resulting from the differential with respect to r and ϕ, respectively:

$$d\boldsymbol{r}_r = dr\cos\phi\,\boldsymbol{i} + dr\sin\phi\,\boldsymbol{j} \tag{2.33}$$

$$d\boldsymbol{r}_\phi = -r\sin\phi\,d\phi\,\boldsymbol{i} + r\cos\phi\,d\phi\,\boldsymbol{j}\,. \tag{2.34}$$

These vectors are orthogonal,

$$\begin{aligned} d\boldsymbol{r}_r\cdot d\boldsymbol{r}_\phi &= (dr\cos\phi\,\boldsymbol{i} + dr\sin\phi\,\boldsymbol{j})\cdot(-r\sin\phi\,d\phi\,\boldsymbol{i} + r\cos\phi\,d\phi\,\boldsymbol{j}) \\ &= -r\,dr\cos\phi\sin\phi\,d\phi + r\,dr\cos\phi\sin\phi\,d\phi \\ &= 0\,, \end{aligned} \tag{2.35}$$

so the area defined by these vectors is obtained by multiplying their magnitudes:

$$\boxed{dA = |d\boldsymbol{r}_r|\cdot|d\boldsymbol{r}_\phi| = dr\cdot r\,d\phi = r\,dr\,d\phi\,.} \tag{2.36}$$

This procedure can be generalized to any coordinate transformation given in terms of new variables u and v: $x = x(u,v)$ and $y = y(u,v)$. The vector $\boldsymbol{r}$ is

$$\boldsymbol{r} = x(u,v)\,\boldsymbol{i} + y(u,v)\,\boldsymbol{j}\,, \tag{2.37}$$

and the differential changes to u and v yield the vectors

$$\boldsymbol{r}_u = \frac{\partial x}{\partial u}\,du\,\boldsymbol{i} + \frac{\partial y}{\partial u}\,du\,\boldsymbol{j} \tag{2.38}$$

$$\boldsymbol{r}_v = \frac{\partial x}{\partial v}\,dv\,\boldsymbol{i} + \frac{\partial y}{\partial v}\,dv\,\boldsymbol{j}\,. \tag{2.39}$$

Since these vectors are not necessarily orthogonal, we must calculate the area from their cross product, which we write as

$$\boldsymbol{r}_u\times\boldsymbol{r}_v = \begin{vmatrix} \dfrac{\partial x}{\partial u}\,du & \dfrac{\partial y}{\partial u}\,du \\ \dfrac{\partial x}{\partial v}\,dv & \dfrac{\partial y}{\partial v}\,dv \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix}\,du\,dv \tag{2.40}$$

The determinant on the right-hand side of this equation is called the **Jacobian** and denoted by $J(u,v)$:

$$\boxed{J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\[2ex] \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix},} \tag{2.41}$$

so the integration element is written as

$$\boxed{dA = |J(u,v)|\, du\, dv\,,} \tag{2.42}$$

where the absolute value is taken because dA is an inherently *positive* quantity, while the sign of J can be changed simple by interchanging the two vectors in the cross product. The Jacobian provides the weight of the integration elements as a function of position. In the case of circular polar coordinates, the Jacobian factor indicates that the weight associated with an element $dr\, d\phi$ increases linearly with r:

$$J(r,\phi) = \begin{vmatrix} \cos\phi & \sin\phi \\[2ex] -r\sin\phi & r\cos\phi \end{vmatrix} = r(\cos^2\phi + \sin^2\phi) = r\,, \tag{2.43}$$

as in Eq. (2.36).

2.2.3 Double Integrals in Polar Coordinates

We first consider the example at the end of Sec. 2.1. The integral is given in Eq. (2.23)

$$\iint_A e^{-x^2-y^2}\, dx\, dy\,,$$

where the region A is shown in Fig. 2.5. In circular polar coordinates in Eq. (2.28), the integrand becomes

$$e^{-x^2-y^2} = e^{-r^2}\,, \tag{2.44}$$

and the integration element becomes $r\,dr\,d\phi$. There remains only the specification of the ranges of r and ϕ. For the semi-circular region in Fig. 2.5,

$$0 \le r \le 1\,, \qquad 0 \le \phi \le \pi\,, \tag{2.45}$$

where the restriction on the range of ϕ results from the fact that the integration region is the *upper* half-circle. The integral to be evaluated is

$$\int_0^{2\pi} d\phi \int_0^1 re^{-r^2}\,dr\,. \tag{2.46}$$

Both integrals are straightforward to carry out, and we obtain

$$\pi \int_0^1 re^{-r^2}\,dr = 2\pi \left(-\tfrac{1}{2}e^{-r^2}\Big|_0^1\right) = \tfrac{1}{2}\pi\left(1 - e^{-1}\right)\,. \tag{2.47}$$

Thus, the transformation into circular polar coordinates has enabled us to evaluate an integral that was intractable in Cartesian coordinates.

Example. Consider the integral of $f(x,y) = 2x + 4y^2$ between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. In circular polar coordinates Eq. (2.28), f is

$$2x + 4y^2 = 2r\cos\phi + 4r^2\sin^2\phi\,. \tag{2.48}$$

The range r is restricted by the radii of the bounding circles, $1 \le r \le 2$, and the range of ϕ is $0 \le \phi < 2\pi$ to account for the entire region between the circles. Hence, the integral to be evaluated is

$$\begin{aligned}
&\int_1^2 r\,dr \int_0^{2\pi} d\phi(2r\cos\phi + 4r^2\sin^2\phi)\\
&= 2\int_1^2 r^2\,dr \underbrace{\int_0^{2\pi} \cos\phi\,d\phi}_{=\,0} + 4\int_1^2 r^3\,dr \underbrace{\int_0^{2\pi} \sin^2\phi\,d\phi}_{=\,\pi}\\
&= 4\pi \int_1^2 r^3\,dr\\
&= \pi r^4\Big|_1^2 = 15\pi\,.
\end{aligned} \tag{2.49}$$

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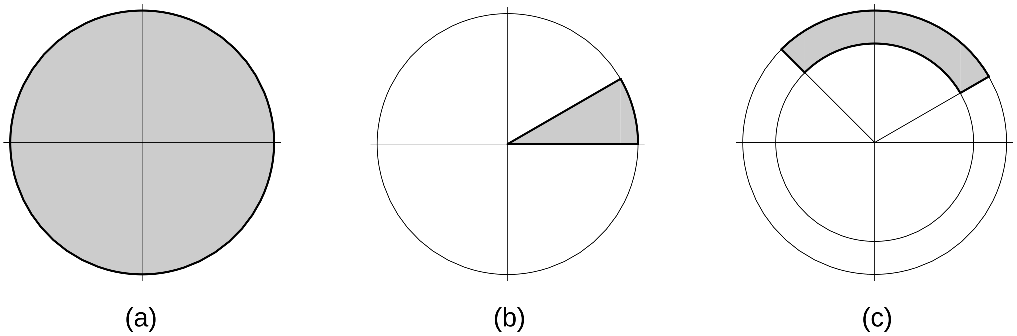


Figure 2.10: The most common types of region, shown shaded, used for integrations in circular polar coordinates.

The most common regions over which integrations are carried out in circular polar coordinates are shown in Fig. 2.10. Figure 2.10(a) represents the interior of a circle of radius R. The ranges of r and ϕ are

$$0 \le r \le R\,, \qquad 0 \le \phi < 2\pi\,. \tag{2.50}$$

For the interior of the wedge-shaped region in Fig. 2.10(b), we have

$$0 \le r \le R\,, \qquad 0 \le \phi \le \Phi\,, \tag{2.51}$$

where Φ is the angle of the wedge. Finally, for the region in Fig. 2.10(c), which is a partial annular region,

$$R_1 \le r \le R_2\,, \qquad \Phi_1 \le \phi \le \Phi_2\,. \tag{2.52}$$

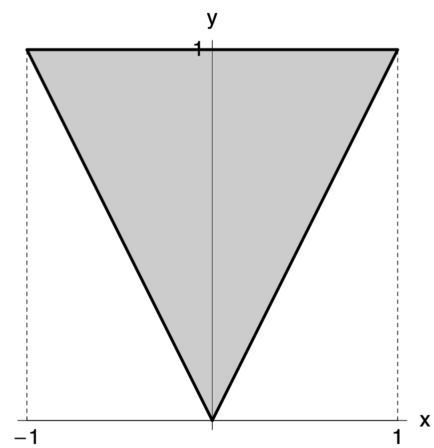
Integrals over all of these regions can therefore be written as

$$\boxed{\iint f(x,y)\,dx\,dy = \int_{R_1}^{R_2}\int_{\Phi_1}^{\Phi_2} f(r\cos\phi, r\sin\phi)\,r\,dr\,d\phi\,.} \tag{2.53}$$

Although regions of the type in Fig. 2.10 are the most natural for circular polar coordinates, the following example shows that integrals over areas with straight boundaries can also be carried out in this coordinate system.

Example.* Consider the area in the figure at right. The azimuthal angle is seen to range between $\tan^{-1}(1) = \frac{1}{4}\pi$ and $\tan^{-1}(-1) = \frac{3}{4}\pi$:

$$\tfrac{1}{4}\pi \le \phi \le \tfrac{3}{4}\pi\,. \qquad (2.54)$$



The lower bound of r is $r = 0$. The upper bound is determined by writing the upper boundary of the triangle, $y = 1$, in circular polar coordinates. Given that $y = r\sin\phi$, we have that this boundary is $r\sin\phi = 1$. The range of r is therefore given by

$$0 \le r \le \frac{1}{\sin\phi}\,, \qquad (2.55)$$

and the area integral is

$$\int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi} d\phi \int_0^{1/\sin\phi} r\,dr\,. \qquad (2.56)$$

The radial integration must be carried out first, and we obtain

$$\int_0^{1/\sin\phi} r\,dr = \tfrac{1}{2}r^2\Big|_0^{1/\sin\phi} = \frac{1}{2\sin^2\phi}\,, \qquad (2.57)$$

and the area integral reduces to

$$\frac{1}{2}\int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi} \frac{d\phi}{\sin^2\phi}\,. \qquad (2.58)$$

The primitive function of the integrand is $-\cot x$:

$$-\frac{d(\cot x)}{dx} = -\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}\,, \qquad (2.59)$$

so the area integral is evaluated as

$$\frac{1}{2}\int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi}\frac{d\phi}{\sin^2\phi}=\frac{1}{2}\left(-\cot x\Big|_{\frac{1}{4}\pi}^{\frac{3}{4}\pi}\right)=\frac{1}{2}(1+1)=1\,. \tag{2.60}$$

which is the area of the triangle. ∎

2.3 Summary

The double integral of a function $f(x,y)$ represents the volume under the surface of that function within a specified region in the x-y plane. This extends to functions of two independent variables the discussion in Sec. 1.2 of one-dimensional integrals. The change of variables from Cartesian to another system of coordinates, such as circular polar coordinates, introduces a term, called the Jacobian, into the integral. The Jacobian is the higher-dimensional analogue of the term obtained by applying the chain rule to the integration element in one-dimensional integrals. It takes into account the changes of the element of integration area across the x-y plane. The evaluation of double integrals proceeds by the successive evaluation of a sequence of one-dimensional integrals.