Chapter 1

Differential and Integral Calculus

There are two basic operations in calculus: differentiation and integration. These are used throughout physics, both as direct operations on physical quantities and for expressing equations that govern the behavior of physical systems. In this section, we review the basic tenets of calculus; the same procedures will be applied throughout this course to vectors in two and three spatial dimensions, so this chapter serves as a template for all of our further discussions.

1.1 Ordinary and Partial Derivatives

The **derivative** of a function f of a single independent variable x is defined by the following limit:

$$\frac{df}{dx} \equiv \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} \right] . \tag{1.1}$$

As the construction in Fig. 1.1 demonstrates, the derivative is the slope of the tangent to f at the point x. The derivative of f is often written as $f'(x)$.

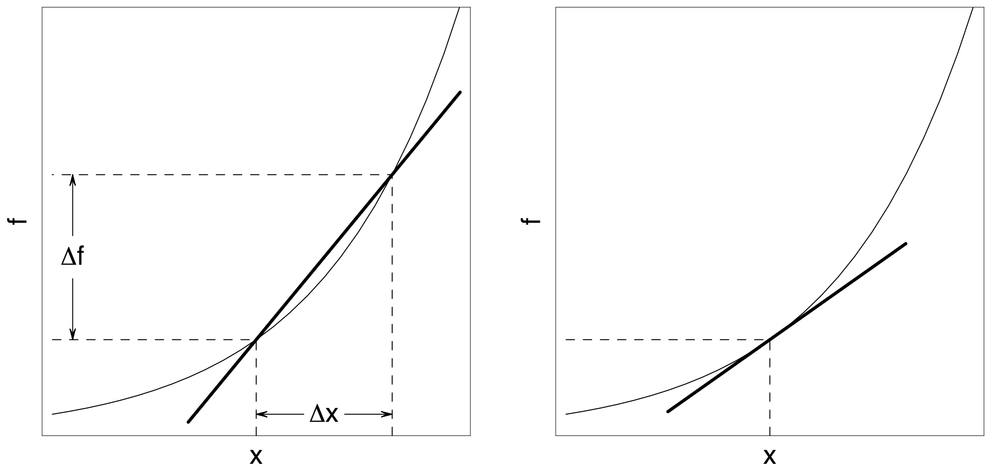


Figure 1.1: The construction of the derivative in Eq. (1.1). The left panel shows a line through the point (x, f) with slope $\Delta f/\Delta x$. The right panel shows the effect of taking the limit $\Delta x \to 0$, which results in a line through (x, f) that is tangent to f at x.

Example. Consider the function $f(x) = x^2$. The derivative of this function with respect to x can be calculated from first principles by using the definition in Eq. (1.1) as follows:

$$\begin{aligned} \frac{d\,x^2}{dx} &= \lim_{\Delta x \to 0} \left[\frac{(x+\Delta x)^2 - x^2}{\Delta x} \right] \\ &= \lim_{\Delta x \to 0} \left[\frac{2x\Delta x + (\Delta x)^2}{\Delta x} \right] \\ &= \lim_{\Delta x \to 0} (2x + \Delta x) \\ &= 2x\,. \end{aligned} \tag{1.2}$$

▮

The basic definition in Eq. (1.1) can be used to show the following well-known formulae of sums, products, and quotients of functions, and the "chain rule" for composite functions (i.e. functions of functions):

$$\frac{d}{dx}(af+bg) = a\frac{df}{dx} + b\frac{dg}{dx}\,, \tag{1.3}$$

$$\frac{d(fg)}{dx} = \frac{df}{dx}g + f\frac{dg}{dx}\,, \tag{1.4}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{1}{g^2}\left(\frac{df}{dx}g - f\frac{dg}{dx}\right)\,, \tag{1.5}$$

$$\frac{d\,f(g(x))}{dx} = \frac{df}{dg}\frac{dg}{dx}\,, \tag{1.6}$$

in which a and b are any constants and f and g are any differentiable constants. Specific derivatives that will be used throughout this course are

$$\frac{d\,x^n}{dx} = nx^{n-1}\,, \tag{1.7}$$

$$\frac{d\,\sin x}{dx} = \cos x\,, \tag{1.8}$$

$$\frac{d\,\cos x}{dx} = -\sin x\,, \tag{1.9}$$

$$\frac{d\,e^{f(x)}}{dx} = \frac{df}{dx}\,e^{f(x)}\,, \tag{1.10}$$

$$\frac{d\,\ln x}{dx} = \frac{1}{x}\,, \tag{1.11}$$

where n is an integer. All of these results will be derived from the definition in Eq. (1.1) in Classwork 1 and Problem Set 1.

The derivative can be extended to functions of more than one variable. For a function f of two independent variables x and y, the **partial derivative**

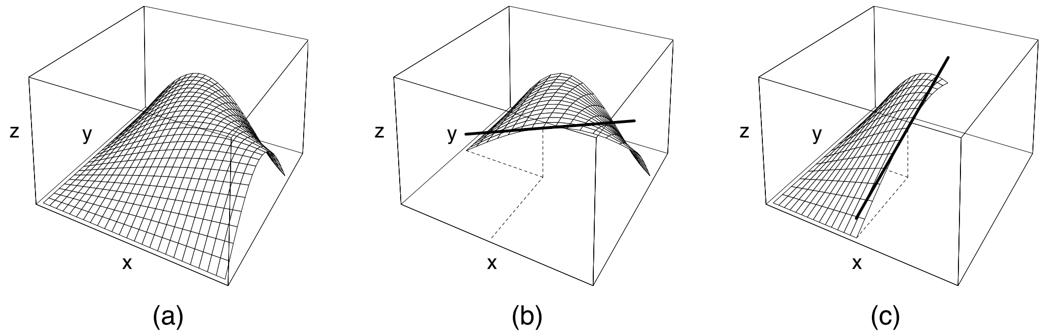


Figure 1.2: (a) A section of a surface $f(x, y)$. (b) The partial derivative of f with respect to x, and (c) the partial derivative of f with respect to y at the same point. The constructions in (b) and (c) show that the two partial derivatives of f can be obtained by slicing the surface parallel to the appropriate axis.

of f with respect to x is defined as (Fig. 1.2)

$$\frac{\partial f}{\partial x} \equiv \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \right] , \tag{1.12}$$

with an analogous expression for the partial derivative $\partial f / \partial y$:

$$\frac{\partial f}{\partial y} \equiv \lim_{\Delta y \to 0} \left[\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \right] . \tag{1.13}$$

As these definitions indicate, when taking the partial derivative with respect to a particular independent variable, the other independent variables are held fixed. Thus, the usual rules of differentiation apply, with these other variables treated effectively as constants. Partial derivatives are often abbreviated with a subscript to indicate the independent variable used for the derivative. In this notation, the two derivatives in Eqs. (1.12) and (1.13) are written as f_x and f_y, respectively. Similarly, the three second-order derivatives are written as f_{xx}, f_{xy}, and f_{yy}. The generalization of partial derivatives to any number of independent variables is straightforward.

The derivative can also be applied to vectors. Consider the quantity

$$\boldsymbol{r}(t) = x(t)\, \boldsymbol{i} + y(t)\, \boldsymbol{j} + z(t)\, \boldsymbol{k} , \tag{1.14}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are the usual unit vectors along the x, y, and z directions, respectively. This may be imagined as the position of a particle in space at time t. The derivative of $\boldsymbol{r}$ with respect to t, which is the instantaneous velocity of the particle, is given by

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\,\boldsymbol{i} + \frac{dy}{dt}\,\boldsymbol{j} + \frac{dz}{dt}\,\boldsymbol{k}\,. \tag{1.15}$$

This vector is tangent to $\boldsymbol{r}$, with a magnitude that is equal to the speed of the particle.

1.2 Integration

The **integral** of a function $f(x)$ over an interval $a \le x \le b$ is defined as the limit of "Riemann sums", which are an approximation to the area bounded by f, the x-axis, and the lines $x = a$ and $x = b$, as indicated in Fig. 1.3. A Riemann sum is constructed by first dividing the interval into N subintervals of length $\Delta x_N \equiv (b-a)/N$. Associated with each subinterval is a strip of area $f(a+n\Delta x_N)\Delta x_N$. The Riemann sum is obtained by adding all of these areas together. The integral of f over this interval is obtained as the limiting value of this sum as the length of the subintervals vanishes ($N \to \infty$):

$$\int_a^b f(x)\,dx \equiv \lim_{N\to\infty}\left[\sum_{n=1}^{N} f(a+n\Delta x_N)\Delta x_N\right], \tag{1.16}$$

Example. The integral of $f(x) = x$ between $x = a$ and $x = b$ is calculated by first constructing the Riemann sum. For this function, we have that

$$f(a+n\Delta x_N) = a + n\Delta x_N\,, \tag{1.17}$$

so the area corresponding to each strip is $(a+n\Delta x_N)\Delta x_n$. Hence, definition in Eq. (1.16) reduces to

$$\int_a^b x\,dx = \lim_{N\to\infty}\left[\sum_{n=1}^{N}(a+n\Delta x_N)\Delta x_N\right]. \tag{1.18}$$

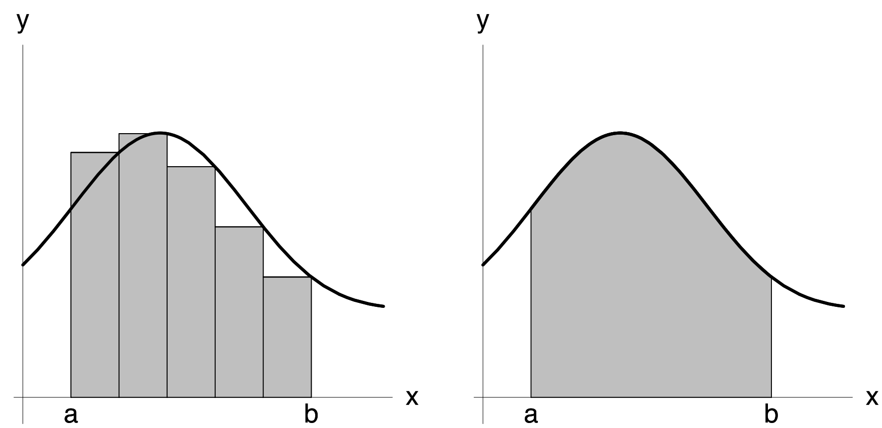


Figure 1.3: The approximation by Riemann sums (left panel) of the area between a curve and the x-axis (right panel).

With $\Delta x_N = (b-a)/N$, we have

$$\begin{aligned}\int_a^b x\,dx &= \lim_{N\to\infty}\left[\sum_{n=1}^{N}\left(a+n\frac{b-a}{N}\right)\left(\frac{b-a}{N}\right)\right]\\ &= \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\left[a\left(\frac{b-a}{N}\right)+n\left(\frac{b-a}{N}\right)^2\right]\right\}. \qquad (1.19)\end{aligned}$$

We can break up the right-hand side of this equation into two separate sums. The first of these can be easily evaluated because there is no explicit n-dependence:

$$\sum_{n=1}^{N} a\left(\frac{b-a}{N}\right) = Na\left(\frac{b-a}{N}\right) = a(b-a)\,. \qquad (1.20)$$

The second sum,

$$\sum_{n=1}^{N} n\left(\frac{b-a}{N}\right)^2 = \left(\frac{b-a}{N}\right)^2\sum_{n=1}^{N} n\,, \qquad (1.21)$$

requires the following result:

$$\sum_{n=1}^{N} n = \tfrac{1}{2}N(N+1)\,. \qquad (1.22)$$

Thus,

$$\left(\frac{b-a}{N}\right)^2 \sum_{n=1}^{N} n = \tfrac{1}{2}(b-a)^2 \frac{N+1}{N}\,. \tag{1.23}$$

Combining these summations and taking the limit $N \to \infty$ allows us to evaluate the integral:

$$\int_a^b x\,dx = a(b-a) + \tfrac{1}{2}(b-a)^2 \underbrace{\lim_{N\to\infty}\left(\frac{N+1}{N}\right)}_{=1} = \tfrac{1}{2}(b^2-a^2)\,. \tag{1.24}$$

$\blacksquare$

1.3 Fundamental Theorem of Calculus

The calculation of an integral as the limit of Riemann sums is much more cumbersome than determining the derivative of a function from Eq. (1.1). Fortunately, the Fundamental Theorem of Calculus alleviates the need such calculations for a large class of integrals. This theorem states that

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)\,,} \tag{1.25}$$

where

$$\frac{dF}{dx} = f\,. \tag{1.26}$$

The function F, whose derivative is equal to f is called the **anti-derivative** or the **primitive function** of f. Note the structure of the Fundamental Theorem. The integral of f is an expression that involves the values of f at every point *within* the interval (a, b). But the evaluation of this integral with the primitive function F of f requires the values of F only at the *endpoints* a and b of this interval. The basic theorems of vector calculus will be seen to have an analogous structure. A proof of the Fundamental Theorem of Calculus is given in the last section of this chapter.

Example. Consider the integral

$$\int_a^b x\,dx\,, \tag{1.27}$$

which was evaluated in the preceding section using Riemann sums. To use the Fundamental Theorem of Calculus, we first identify the primitive function F of x as

$$F(x) = \tfrac{1}{2}x^2 + A\,, \tag{1.28}$$

where A is a constant (called a "constant of integration"). Then, the value of this integral is

$$\int_a^b x\,dx = \left(\tfrac{1}{2}x^2 + A\right)\bigg|_a^b = \tfrac{1}{2}(b^2 - a^2)\,. \tag{1.29}$$

Note that the constant A makes no contribution to the value of the integral. Thus, for the purposes of evaluating *definite* integrals, constants of integration can be omitted from the primitive function F. ∎

The Fundamental Theorem of Calculus enables a number of important properties of integrals to be obtained. Higher-dimensional versions of this theorem form one of the major themes of this course. The following properties of definite integrals are implied by the Fundamental Theorem:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\,, \tag{1.30}$$

$$\frac{d}{dx}\left[\int_a^x f(s)\,ds\right] = f(x)\,, \tag{1.31}$$

$$\frac{d}{dx}\left[\int_x^b f(s)\,ds\right] = -f(x)\,, \tag{1.32}$$

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(s)\,ds\right] = \frac{dv}{dx}f(v(x)) - \frac{du}{dx}f(u(x))\,. \tag{1.33}$$

1.4 Variable Transformations in Integrals

The fundamental theorem of calculus establishes a connection between derivatives and integrals and provides a way of evaluating integrals in principle. But evaluating the anti-derivative of particular functions (i.e. finding F such that $dF/dx = f$) may still prove a challenging proposition and in some cases may require numerical evaluation (e.g. the trapezoidal method). Several methods have been developed for finding anti-derivatives in particular cases, including integration by parts, trigonometric substitution and other variable transformations, and partial fractions. Variable transformations in particular provide a versatile way of changing difficult integrals into expressions that are more manageable. We first work through an example.

Example. Consider the integral

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}\,. \tag{1.34}$$

This a standard example of an integral whose evaluation benefits from a change of variables, in this case based on trigonometric functions. We define a new variable of integration through

$$x = \sin\theta\,. \tag{1.35}$$

To transform the integral, we must consider the effect of this transformation on the integrand, the integration element, and the limits of integration. By using the trigonometric identity $\cos^2\theta + sin^2\theta = 1$, the integrand is transformed to

$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-\sin^2\theta}} = \frac{1}{\cos\theta}\,. \tag{1.36}$$

The integration element is calculated by applying the chain rule to Eq. (1.35):

$$dx = \cos\theta\,d\theta\,. \tag{1.37}$$

Lastly, the new limits of integration are determined by identifying the values of θ whose values are 0 for the lower limit, and 1 for the upper limit. These are identified as

$$\sin(0) = 0\,, \qquad \sin(\tfrac{1}{2}\pi) = 1\,. \tag{1.38}$$

Thus, the original integral is transformed to

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \int_0^{\pi/2} d\theta\,, \tag{1.39}$$

the right-hand side of which is easily evaluated, and we obtain

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \tfrac{1}{2}\pi\,. \tag{1.40}$$

This example illustrates the power of variable transformations: what seemed as difficult evaluation has been transformed, through a judicious choice of a new integration variable, to a much simpler expression. An appropriate transformation is sometimes apparent from the integral itself, as in this example, but may involve an element of trial and error. Most modern computational mathematics software, e.g *Maple* and *Mathematica*, perform a series of transformations to determine the simplest form of an integral. ∎

We can now formulate in general terms the transformation of an integral

$$\int_a^b f(x)\,dx \tag{1.41}$$

under the change of variables $x \to t(x)$. The integrand becomes

$$f(x) = f(x(t))\,, \tag{1.42}$$

where $x(t)$ is obtained from the inverse of $t(x)$. Note that in the example above,, the change of variables was defined in this form. The integration element is transformed to

$$dx = \frac{dx}{dt}\,dt \tag{1.43}$$

and the limits of integration are now

$$t(a) \qquad \text{and} \qquad t(b)\,. \tag{1.44}$$

Hence, the general form of a change of variables in an integral is

$$\boxed{\int_a^b f(x)\,dx = \int_{t(a)}^{t(b)} f(x(t))\,\frac{dx}{dt}\,dt\,.} \tag{1.45}$$

The choice of transformation is usually dictated by the requirement that the primitive function of the transformed integrand, $f(x(t))(dx/dt)$, is easier to determine than the original function. The quantity dx/dt represents the change in the density of integration points induced by the change of variables. This is a key quantity that arises whenever integration variables are changed and will be encountered again when we discuss coordinate transformations in two and three dimensions.

The following indefinite integrals, which can be derived from the basic formulae in Eqs. (1.7)–(1.11), will be used throughout this course:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C\,, \tag{1.46}$$

$$\int \sin x\,dx = -\cos x + C\,, \tag{1.47}$$

$$\int \cos x\,dx = \sin x + C\,, \tag{1.48}$$

$$\int e^{\pm x}\,dx = \pm e^{\pm x} + C\,, \tag{1.49}$$

$$\int \ln x\,dx = \frac{1}{x} + C\,, \tag{1.50}$$

$$\int \cos^2 x\,dx = \tfrac{1}{2}x + \tfrac{1}{2}\sin x\cos x + C\,, \tag{1.51}$$

$$\int \sin x\cos^n x\,dx = -\frac{1}{n+1}\cos^{n+1} x + C\,, \tag{1.52}$$

$$\int x^2\,e^{-x}\,dx = -(x^2 + 2x + 2)\,e^{-x} + C\,. \tag{1.53}$$

where n is a positive integer and C is a constant that is eliminated once the integrals are evaluated between specific upper and lower limits. As guaranteed by the Fundamental Theorem, the derivative of the right-hand side of each of these expressions yields the integrand on the corresponding left-hand side.

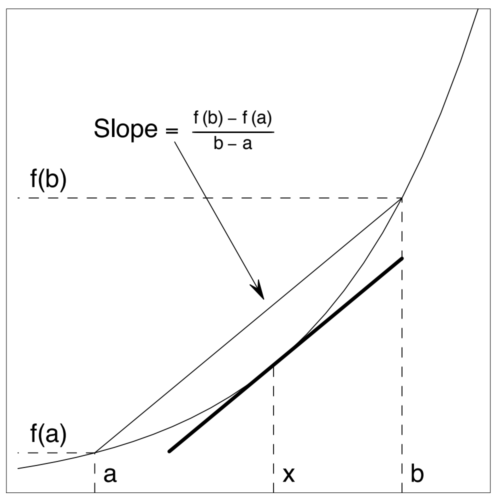
1.5 Proof of the Fundamental Theorem*

Proving the Fundamental Theorem of Calculus requires that we first prove the **Mean Value Theorem**:

If f is a real continuous function on an interval $[a, b]$ and differentiable on the open interval (a, b), then there is a point x within (a, b) at which

$$f(b) - f(a) = (b - a) f'(x) \, .$$

This theorem is straightforward to understand in terms of the diagram shown below.



The quantity

$$\frac{f(b) - f(a)}{b - a}$$

represents the slope of the straight line passing through the end-points $(a, f(a))$ and $(b, f(b))$. The Mean Value Theorem states that, if f is differentiable everywhere within (a, b), there is a point x within this interval where the slope $f'(x)$ of f is given by

$$f'(x) = \frac{f(b) - f(a)}{b - a} \, .$$

This is shown by the emboldened line in the figure above.

To use the Mean Value Theorem to prove the Fundamental Theorem of Calculus, we define the function F by

$$F(x) = \int_a^x f(t)\,dt$$

for some function f and $a \le x \le b$. We will first show that F is a differentiable function where f is continuous. Using the definition in Eq. (1.1), we write the derivative of F as

$$\begin{aligned}
\frac{dF}{dx} &= \lim_{\Delta x\to 0}\left[\frac{F(x+\Delta x)-F(x)}{\Delta x}\right] \\
&= \lim_{\Delta x\to 0}\left\{\frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt\right]\right\} \\
&= \lim_{\Delta x\to 0}\left\{\frac{1}{\Delta x}\left[\int_a^x f(t)\,dt + \int_x^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt\right]\right\} \\
&= \lim_{\Delta x\to 0}\left[\frac{1}{\Delta x}\int_x^{x+\Delta x} f(t)\,dt\right].
\end{aligned}$$

The integral in the last line of this equation can be approximated by the area of a strip of height $f(x)$ and width Δx, with a correction of order $(\Delta x)^2$:

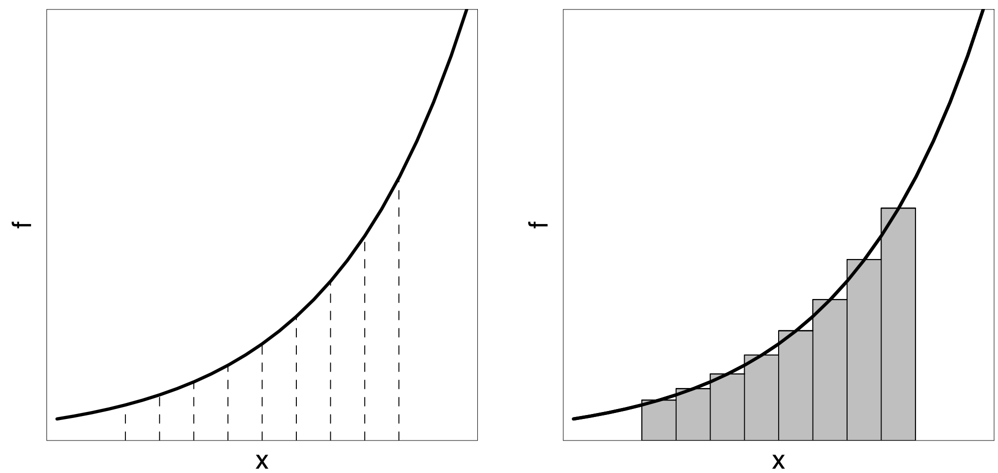
$$\int_x^{x+\Delta x} f(t)\,dt = f(x)\Delta x + O(\Delta x)^2\,.$$

Hence, upon substitution of this expression into the definition of the derivative of F, we obtain

$$\begin{aligned}
\frac{dF}{dx} &= \lim_{\Delta x\to 0}\left\{\frac{1}{\Delta x}\Big[f(x)\Delta x + O(\Delta x^2)\Big]\right\} \\
&= \lim_{\Delta x\to 0}\,[f(x) + O(\Delta x)] \\
&= f(x)\,,
\end{aligned}$$

which demonstrates that the derivative of F exists for every point x where f is continuous. In particular, if f is a continuous function on $[a, b]$, then F

is differentiable at every point in that interval. Thus, consider the partition of $[a, b]$ into N intervals such that $x_{n-1} \leq x \leq x_n$, where $x_0 = a$ and $x_N = b$, as shown below:



We now use the Mean Value Theorem to choose a point t_n within the nth interval that satisfies

$$F(x_n) - F(x_{n-1}) = (x_n - x_{n-1})F'(t_n) = (x_n - x_{n-1})f(n_i) \,.$$

Then

$$F(b) - F(a) = \sum_{n=1}^{N} [F(x_n) - F(x_{n-1})] = \sum_{n=1}^{N} f(t_n)\Delta x_n \,,$$

where $\Delta x_n = x_n - x_{n-1}$. The right-hand-side of this equation is represented by the shaded area in the right panel in the figure above and is seen to be the same basic construction as that used for Riemann sums shown in the figure above. Accordingly, if we now take the limit $N \to \infty$ this approaches the area under the curve, and we have

$$F(b) - F(a) = \int_a^b f(t)\, dt \,,$$

which is the Fundamental Theorem of Calculus.