# Imperial College London

# BSc/MSci EXAMINATION June 2013

*This paper is also taken for the relevant Examination for the Associateship*

# MATHEMATICS AND STATISTICS OF MEASUREMENT

## For Second-Year Physics Students

Friday, 7th June 2013: 10:00 to 12:00

*Answer all questions*
*Marks shown on this paper are indicative of those the Examiners anticipate assigning.*

**General Instructions**

Complete the front cover of each of the 4 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 4 answer books even if they have not all been used.

**You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.**

**1.** (i) Write down the general expression for the Fourier series of a function $f(x)$ of period $2L$, and the Euler-Fourier formulae for the coefficients $a_0$, $a_n$, $b_n$. Determine the Fourier series for the periodic function $f_1(x)$, period $2L$, defined by:

$$f_1(x) = \begin{cases} A - x, & -L < x < 0 \\ A + x, & 0 < x < L \end{cases}$$

where $A$ is a constant. [9 marks]

(ii) Using the notation that the Fourier Transform of $f(x)$ is defined by:

$$\mathcal{F}[f(x)] = g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \ ,$$

compute the Fourier transform of the function $f_2(x)$ defined by:

$$f_2(x) = \begin{cases} 0, & -\infty < x < 0 \\ x, & 0 < x < L \\ 0, & L < x < \infty \end{cases}$$

[5 marks]

(iii) Defining $h(x)$ as the convolution of $f_2(x)$ with itself, i.e. $h(x) = f_2(x) * f_2(x)$, show that $h(x)$ is given by:

$$h(x) = \begin{cases} 0, & -\infty < x < 0 \\ \frac{x^3}{6}, & 0 < x < L \\ -\frac{2L^3}{3} + L^2 x - \frac{x^3}{6}, & L < x < 2L \\ 0, & 2L < x < \infty \end{cases}$$

State the convolution theorem, and explain its advantage in the case that it were necessary to compute the Fourier transform of $h(x)$. [11 marks]

[Total 25 marks]

**2.** (i) Solve the following initial value problems

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)} \qquad y(0) = -1$$

$$\frac{dy}{dt} + \frac{2}{t}y = \frac{\cos t}{t^2} \qquad y(\pi) = 0\ .$$

[5 marks]

(ii) Laguerre's differential equation is:

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + my = 0\ ,$$

where $m$ is a constant, not necessarily an integer. Using the Frobenius method, show that the equation has one series solution of the form:

$$y(x) = C_0 \sum_{n=0}^{\infty} \frac{(-1)^n m(m-1)(m-2)...(m-n+1)}{(n!)^2} x^n,$$

where $C_0$ is an arbitrary constant. [10 marks]

(iii) What is the radius of convergence of the series? [5 marks]

(iv) Show that the series terminates when $m$ is an integer. In such a case the solution may be written as:

$$y(x) = C_0 L_m(x)$$

where $L_m(x)$ are Laguerre's polynomials. The first two Laguerre polynomials are given by:

$$L_0(x) = 1, \qquad L_1(x) = 1 - x\ .$$

Verify that $L_0$ and $L_1$ are orthogonal on the interval $[0, \infty]$ with respect to the weight function $e^{-x}$, by evaluating the integral

$$\int_0^{\infty} e^{-x} L_0 L_1 \, dx\ .$$

[5 marks]

[Total 25 marks]

**3.** (i) Determine the general solution of the following differential equation:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 10 \ .$$

Also confirm that the solutions you find for the homogeneous equation are independent. [5 marks]

(ii) The temperature in a two-dimensional slab is determined by the solution of Laplace's equation in two dimensions:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \ .$$

By seeking solutions of the form $T = X(x)Y(y)$, show that Laplace's equation is separable and takes the form:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} \ .$$

If the separation constant is $-k^2$, derive the general solutions for $X$ and $Y$.

[6 marks]

(iii) Consider the steady-state temperature in a two-dimensional slab with boundaries at $x = 0$, $x = a$, $y = 0$. The slab extends to infinity in the $+y$ direction. The boundary conditions are: (a) $T = 0$ at $x = 0$ and (b) $T = 0$ at $x = a$, for all values of $y$, (c) $T \to 0$ as $y \to \infty$, (d) $T = T_0$ at $y = 0$, for all values of $x$. Why are the values of the separation constant $k^2 = 0$ and $k^2 > 0$ not appropriate for this problem? [Include equations in your answers as required.] [5 marks]

(iv) By applying the first three boundary conditions. show that the solution is:

$$T(x, y) = B_n \sin(n\pi x/a) \exp(-n\pi y/a) \ ,$$

where $n$ is an integer and the $B_n$ are constants. [4 marks]

(v) Show that enforcement of the final boundary condition gives the solution:

$$T = \frac{4T_0}{\pi} \sum_{n\,odd} \frac{1}{n} \sin(n\pi x/a) \exp(-n\pi y/a) \ .$$

[5 marks]

[Total 25 marks]

**4.** (i) A bent coin has a probability $p = 0.40$ of landing tails. It is thrown $N$ times.

(a) For $N = 10$, what is the probability of obtaining 3 tails? [2 marks]

(b) What is the expectation value and variance for the number of tails for $N = 10$? [2 marks]

The coin is now thrown $N = 16$ times and $T = 6$ tails are observed.

(c) What is the likelihood function for this measurement? Identify clearly the parameter and the data. [3 marks]

(d) Derive the maximum likelihood estimate for the value of the probability of tails, $p$, and show that it is given by $p_{\mathrm{ML}} = T/N$ [4 marks]

(e) By using a Gaussian approximation to the likelihood, show that the 1 sigma uncertainty in the maximum likelihood estimate for $p$ is given by

$$\Sigma = \left( \frac{N}{\frac{T}{N}\left(1 - \frac{T}{N}\right)} \right)^{-1/2} .$$

[4 marks]

(f) Given the above result, estimate the number of sigma confidence with which the hypothesis that $p = 0.40$ can be excluded by the above measurement. [2 marks]

(g) If you wanted to exclude the hypothesis that $p = 0.40$ with at least 5 sigma confidence, how many measurements should you make? Assume that the numerical value of $p_{\mathrm{ML}} = T/N$ remains constant as $N$ increases. [3 marks]

(ii) State Bayes theorem as applied to the problem of inference for a parameter $\theta$ from data $d$, clearly identifying each term and explaining their meaning. [5 marks]

[Total 25 marks]