Imperial College London

BSc/MSci EXAMINATION May 2012

This paper is also taken for the relevant Examination for the Associateship

MATHEMATICS AND STATISTICS OF MEASUREMENT

For Second-Year Physics Students

Thursday, 31st May 2012: 14:00 to 16:00

Answer ALL parts of Section A and TWO parts of Section B.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

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SECTION A

1. (i) The Fourier transform of $f(x)$ is

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \ .$$

Denoting the Fourier transform of $f(x)$ by $\mathcal{F}[f(x)]$, prove the *scaling* relationship

$$\mathcal{F}[f(ax)] = \frac{1}{|a|} g\left(\frac{\omega}{a}\right) ,$$

where a is a constant.

[3 marks]

(ii) Solve the initial value problem

$$t\frac{dy}{dt} + 2y = 4t^2 \qquad y(1) = 4 ,$$

by means of an integrating factor. [3 marks]

(iii) Determine the general solution of the following differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 12 \ .$$

Also confirm that the solutions you find for the homogeneous equation are independent. [4 marks]

[Total 10 marks]

2. (i) A biased coin has a probability θ of landing heads. It is tossed $N > 2$ times. What is the probability of obtaining $N - 2$ heads? For $\theta = 0.9$ and $N = 10$, evaluate the probability of obtaining 9 or more heads. [3 marks]

(ii) A fair die with 6 sides is rolled once. What is the expectation value for the number of dots, f? What is its variance?

The same die is now rolled $N = 100$ times and the rolled values are added up together. Find the expectation value and standard deviation of the total sum.

[3 marks]

(iii) The current in a wire is measured N times with a device subject to Gaussian noise. If one wanted to double the precision of the measurement of the current, how many measurements should one make in total? Justify your answer. You may assume that each measurement is independent. [2 marks]

(iv) State the Maximum Likelihood Principle and explain how it is used to determine the value of a parameter. [2 marks]

[Total 10 marks]

SECTION B

3. A generator produces a voltage

$$V(t) = A\,|\sin(2\pi f t)|$$

with $f = 60$Hz and $A = 100$ volts.

(i) Sketch the function $V(t)$ for both positive and negative time, for a wide enough range in t that the behaviour of the function is obvious. Be sure to label the axes. [3 marks]

(ii) What is its period, T? (Specifically, what is the *smallest* value you could define as the period?) [1 mark]

(iii) Is the function odd, even, or neither? [1 mark]

(iv) Evaluate the quantity

$$P = \frac{1}{T}\int_0^T |V(t)|^2\, dt\ .$$

[3 marks]

(v) Calculate the coefficients of the Fourier series a_n, b_n to show that we can write

$$V(t) = \frac{2A}{\pi} + \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\cos(4\pi n f t)}{1-4n^2}$$

[9 marks]

(vi) Parseval's Theorem states that $P = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$. Using this and the previous answers, or otherwise, show that

$$\sum_{n=-\infty}^{\infty}\frac{1}{(1-4n^2)^2} = \frac{\pi^2}{8}$$

[3 marks]

[Total 20 marks]

4. The differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right)y = 0 \ ,$$

has a regular singular point at $x = 0$. (This is Bessel's equation of order $\frac{1}{2}$.)

(i) Show that the Frobenius series

$$y(x) = \sum_{n=0}^{\infty} a_n x^{s+n}$$

satisfies the differential equation if

$$a_0\left(s^2 - \frac{1}{4}\right)x^s + a_1\left[(s+1)^2 - \frac{1}{4}\right]x^{s+1} + \sum_{n=2}^{\infty}\left[a_n\left((s+n)^2 - \frac{1}{4}\right) + a_{n-2}\right]x^{s+n} = 0 \ .$$

[8 marks]

(ii) Write down the indicial equation and show that $s = \pm\frac{1}{2}$. [2 marks]

(iii) Considering the case when $s = +\frac{1}{2}$, show that

(a) a_0 is arbitrary;

(b) $a_1 = 0$;

(c) $a_n = -\frac{a_{n-2}}{n(n+1)}$ when $n \geq 2$.

Hence show that the series solution for this case is given by

$$y = a_0\frac{\sin x}{\sqrt{x}} \ .$$

[9 marks]

(iv) For the $s = -\frac{1}{2}$ case, the series solution is:

$$y = a_0\frac{\cos x}{\sqrt{x}} + a_1\frac{\sin x}{\sqrt{x}} \ .$$

What is the general solution to the differential equation? [1 mark]

[Total 20 marks]

5. A string is stretched tightly and extended between two points $x = 0$ and $x = L$. The vertical displacement $u(x, t)$ is governed by the wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad 0 \le x \le L \ ,$$

where c is the wave velocity.

(i) Show that assuming a separable solution of the form $u(x, t) = X(x)T(t)$, leads to the following ordinary differential equations satisfied by $X(x)$ and $T(t)$;

$$\frac{d^2X}{dx^2} = SX, \quad \frac{d^2T}{dt^2} = Sc^2T \ ,$$

where S is a separation constant. [4 marks]

(ii) Assuming that $S = -k^2$ is negative, show that the boundary conditions impose a general solution of the form:

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right] .$$

[7 marks]

(iii) At $t = 0$,

$$u(x, 0) = \sin\left(\frac{\pi x}{L}\right) ,$$

$$\frac{\partial u(x, 0)}{\partial t} = 0 .$$

Determine expressions for the coefficients A_n, B_n of the solution in (ii), where you may assume $B_n = 0$ for all values of $n \neq 1$. [7 marks]

(iv) Write down the complete solution for $u(x, t)$ taking into account the initial conditions in (iii). What type of wave has resulted? [2 marks]

[Total 20 marks]

6. A coin is tossed N times to determine whether it is fair or not, landing heads H times.

(i) Write down the likelihood function for the probability of the coin landing heads, θ, clearly identifying the parameter and the data. Show that the Maximum Likelihood estimation for the probability of the coin landing heads, θ, is given by

$$\theta_{\mathrm{ML}} = \frac{H}{N}. \tag{1}$$

[3 marks]

(ii) By using a Gaussian approximation to the likelihood, show that the 68.3% symmetric confidence interval for θ is given by

$$\Sigma_\theta = \frac{\left[\frac{H}{N}\left(1 - \frac{H}{N}\right)\right]^{1/2}}{\sqrt{N}}. \tag{2}$$

For $N = 250$ and $H = 149$, determine the 95% confidence interval for θ. On this basis, can you conclude that, at the 95% confidence level, the hypothesis of a fair coin can be rejected? Justify your answer. [6 marks]

(iii) We now wish to analyze this situation using Bayes theorem instead. Write down Bayes theorem, and clearly identify, and name, each component of the equation. Explain what each means. [4 marks]

(iv) Let us assume a uniform prior for θ, i.e.

$$P(\theta) = \left\{ \begin{array}{ll} 1, & \text{if } 0 \leq \theta \leq 1 \\ 0, & \text{otherwise.} \end{array} \right\} \tag{3}$$

Using a Gaussian approximation to the likelihood, find the posterior probability distribution function for θ. [3 marks]

(v) Show that the posterior mean for θ,

$$\langle\theta\rangle = \int_0^1 P(\theta|d)\theta d\theta \tag{4}$$

(where $P(\theta|d)$ is the posterior probability distribution found above, and d are the data) coincides with the MLE of Eq. (1).

Find the symmetric interval around the posterior mean containing 99% of posterior probability for θ. [4 marks]

[This question continues on the next page . . .]

The relationship between the size of the interval around the mean and the probability content for a Gaussian distribution is given below. [Total 20 marks]

κ "number of sigma"	$P(-\kappa < \frac{x-\mu}{\sigma} < \kappa)$ Probability content	Usually called
1	0.683	1σ
2	0.954	2σ
3	0.997	3σ
4	0.9993	4σ
5	$1 - 5.7 \times 10^{-7}$	5σ
1.64	0.90	90% probability interval
1.96	0.95	95% probability interval
2.57	0.99	99% probability interval
3.29	0.999	99.9% probability interval