Fact Sheet M – The Eigenvalue Problem

The *eigenvalue problem* is defined by the equation

$$\mathbf{Tr} = \lambda\mathbf{r} \qquad (1)$$

where **T** is a square matrix, and **r** is a column matrix. One is seeking vectors **r** (known as *eigenvectors*) that are unchanged in direction by **T**, but are simply scaled in magnitude by the factors λ (known as *eigenvalues*). Eq.(1) can be written

$$\mathbf{Tr} = \lambda\mathbf{Ur} \qquad (2)$$

or

$$(\mathbf{T} - \lambda\mathbf{U})\mathbf{r} = 0 \qquad (3)$$

where **U** is the unit matrix.

For a 2×2 system, eq.(1) reads

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix} \qquad (4)$$

and eq.(3) becomes

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \qquad (5)$$

If you are unconvinced by eq.(5), try writing out eq.(4) in full!

Eq.(5) represents the two homogeneous linear equations

$$\begin{aligned} (a-\lambda)x + by &= 0 \\ cx + (d-\lambda)y &= 0 \end{aligned} \qquad (6)$$

These equations have a solution (other than the trivial solution $x = y = 0$) if *and only if* the determinant of the coefficients is zero i.e.

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = 0 \qquad (7)$$

or in other words

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0\,. \qquad (8)$$

The two roots of this quadratic equation are

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2} \qquad (9)$$

and are the two eigenvalues of **T**.

In the case treated in Lecture 15, $\mathbf{T} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$ and $\lambda_{1,2} = \frac{10 \pm 2}{2} = 6 \text{ or } 4$. These eigenvalues must now be inserted in turn in eqs.(6) to find the eigenvectors. One obtains

$$\lambda_1 = 6; \qquad x_1 + y_1 = 0; \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \qquad \hat{\mathbf{v}}_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \tag{10}$$

$$\lambda_2 = 4; \qquad x_2 - y_2 = 0; \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \qquad \hat{\mathbf{v}}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \tag{11}$$

where the final step in each case gives the eigenvector in normalised form.

Note that eq.(4) can be written

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \tag{12}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lambda_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \tag{13}$$

These last two equations can be combined to read

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{14}$$

or

$$\mathbf{TV = VD} \tag{15}$$

where **V** is the matrix of the eigenvectors $\mathbf{T} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ and **D** is the diagonal matrix of the eigenvalues $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Multiplying both sides of eq.(15) from the left by the inverse of **V** yields

$$\mathbf{V^{-1}TV = D} \tag{16}$$

The matrices **T** and **D** are said to be "similar", and **V** is said to "diagonalise" **T** in a "similarity transformation". Note that eq.(14) applies irrespective of whether the eigenvectors have been normalised before the formation of **V**. However, in the special case where **T** is symmetric (i.e. $b = c$ as in the numerical example given above), it can be shown that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. If the vectors are used in normalised form in this case, then **V** becomes an orthogonal matrix, in which case $\mathbf{V}^{-1} = \mathbf{V}^{\mathbf{T}}$.

In the example above, for instance, $\mathbf{V} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ and $\mathbf{V}^{-1} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$. In this case

$$\mathbf{V^{-1}TV} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix} = \mathbf{D} \tag{17}$$

which checks out nicely.

Note that for a 3×3 system, eq.(8) becomes a cubic equation, for a 4×4 system, it's a quartic and so on. If you are asked to solve a cubic equation in a problem sheet or examination, you can be confident that the roots are small integers, so you only have to spot one of them (λ_1), divide out the factor $(\lambda - \lambda_1)$, and solve the remaining quadratic for the other two roots.