Fact Sheet L – Linear Equation Scenarios

..... when the determinant of the coefficients is zero .....

This fact sheet contains detailed information about the different situations that occur for a set of three linear equations in three unknowns, when the matrix of the coefficients is "singular" and the determinant of the coefficients is zero. Under these circumstances, there is either no solution at all, or the solution is not uniquely defined – it may be a line, or a plane.

At the outset, remember the following facts:

- $ax+by+cz=k$ is the equation of a plane;
- The normal to the plane lies in the direction of the vector $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$;
- The distance from the origin to the plane in the direction of the normal is $\dfrac{k}{\sqrt{a^2+b^2+c^2}}$;
- In looking for the solution of three such equations, we are concerned with the intersection of the planes. <u>Unless the determinant of the coefficients is zero, the three planes intersect at a point, and there is a unique solution</u>.
- When the determinant of the coefficients <u>*is*</u> zero, the normals to the planes are coplanar, which is reflected in the fact that any row of the matrix is a linear superposition of the other two.

The six possible scenarios when the determinant is zero are as follows:

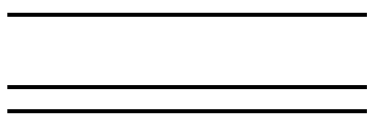
1. For scenario 1, all three planes are the same, so all three equations are the same. Remember however that they may not <u>*appear*</u> to be the same at first sight. An example is

$$\begin{aligned} x+2y-3z&=5\\ -2x-4y+6z&=-10\\ 3x+6y-9z&=15 \end{aligned}$$

Since all the equations say the same thing, any one equation defines the plane solution.

2. For scenario 2, the three planes are parallel, but all are different and so there is no solution. An example is

$$\begin{aligned} x+2y-3z&=5\\ x+2y-3z&=6\\ x+2y-3z&=8 \end{aligned}$$



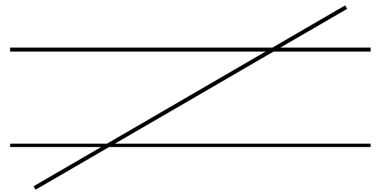
The left-hand sides are all the same, but the right-hand sides are different, making the equations contradictory and hence inconsistent.

There is of course an intermediate case between 1 and 2, in which two planes are coincident and the third is different.

>>>>>>>>>>>>>

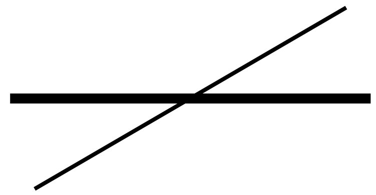
3. This scenario has two parallel planes, and the third inclined; again there is no solution. An example is

$$\begin{aligned} x+2y-3z&=5\\ x+2y-3z&=6\\ x+y+3z&=4 \end{aligned}$$



4. Here the two parallel planes of the previous scenario are coincident, and there is a line solution. An example is

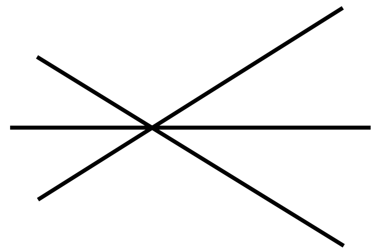
$$\begin{aligned} x+2y-3z&=5\\ -2x-4y+6z&=-10\\ x+y+3z&=4 \end{aligned}$$



Since the first two equations say the same thing, one can be discarded to find the equation of the line.

5. In this scenario, three different planes intersect on a common line. An example is

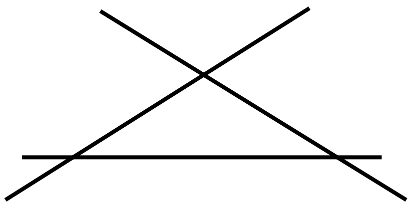
$$\begin{aligned} x+2y-3z&=5\\ x+y+3z&=4\\ x+3y-9z&=6 \end{aligned}$$



The third equation is twice the first minus the second. Any two equations will yield the line solution.

6. The previous scenario is a special case of the “toblerone” situation of this final scenario. Here, three planes intersect in three different parallel lines, and there is no solution of any kind. An example is

$$\begin{aligned} x+2y-3z&=5\\ x+y+3z&=4\\ x+3y-9z&=7 \end{aligned}$$



The only difference between these equations and those in scenario 5 is that the number on the rhs of the third equation has been changed.