Fact Sheet I – Matrices

- A *matrix* is a rectangular pattern of numbers that has specific mathematical properties, and obeys specific mathematical rules.
- An $r \times s$ matrix has *r* rows and *s* columns. A 2×3 matrix with 2 rows and 3 columns looks like

  $$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

  Other notations are often used including *double suffix notation* in which the *matrix elements* are written a_{jk} where *j* is the row number and *k* the column number i.e.

  $$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

- A matrix with only one row ($r = 1$) is called a *row matrix*, while a matrix with only one column ($s = 1$) is called a *column matrix*. The terms *row vector* and *column vector* are sometimes used, because the elements can be regarded as the components of a vector.
- Two matrices can be added or subtracted only if they have the same number of tows and columns. Simply add or subtract corresponding elements in the two matrices.
- The *matrix multiplication rule* for two 2×2 matrices is

  $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

- In double suffix notation, the matrix multiplication rule for $\mathbf{C} = \mathbf{AB}$ is

  $$c_{jk} = \sum_m a_{jm} b_{mk}$$

  where the first index is (as always) the row number and the second the column number Note that two matrices can be multiplied if (*and only if*) the number of columns of **A** equals the number of rows of **B**. i.e. if $s_A = r_B$. The resulting matrix **C** is a matrix with r_A rows and s_B columns.

- Matrix multiplication is *non-commutative*, i.e. $\mathbf{AB} \neq \mathbf{BA}$. Indeed it is possible that two matrices can be multiplied in one order but not in the other.
- The *transpose* $\mathbf{A}^{\mathrm{T}}$ of a matrix **A** is obtained by exchanging rows and columns i.e. $a_{jk} \to a^T_{kj}$.
- A *square matrix* has the same number of rows as columns (i.e. $r = s$). A 3×3 matrix looks like

  $$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- The *leading diagonal* of a square matrix **A** runs from top left to bottom right and includes elements with equal indices a_{ii} .
- A *symmetric matrix* is a square matrix that is unchanged if the elements are reflected in the leading diagonal; it follows that a symmetric matrix is equal to its transpose.
- A *diagonal matrix* is a square matrix with non-zero elements only on the leading diagonal.
- A *unit matrix* **U** is a diagonal matrix with all elements on the leading diagonal equal to unity.
- A *determinant* (written $\det\{\mathbf{A}\}$) is properly referred to as *the determinant of a matrix* . Only square matrices have determinants.
- The *minor* M_{jk} of element *jk* of a square matrix is the determinant of the smaller matrix obtained when row *j* and column *k* of the original matrix are ignored.
- The *cofactor* C_{jk} is the so-called "signed minor" defined by $C_{jk} = (-1)^{j+k} M_{jk}$.
- The *adjoint* matrix (written $\mathrm{adj}\{\mathbf{A}\}$) is obtained by writing down the matrix of the cofactors, and taking the transpose.
- A *singular matrix* is a square matrix whose determinant is zero and which therefore has no inverse. (The term is sometimes applied to *any* matrix that has no inverse, which includes all non-square matrices!)
- The *inverse* $\mathbf{A}^{-1}$ of a square matrix **A** is defined such that $\mathbf{A}\mathbf{A}^{-1} \equiv \mathbf{A}^{-1}\mathbf{A} = \mathbf{U}$. The formula for the inverse is $\mathbf{A}^{-1} = \dfrac{\mathrm{adj}\{\mathbf{A}\}}{\det\{\mathbf{A}\}}$. Note that matrices that are not square have no inverse, and nor do square matrices that are *singular*.
- An *orthogonal matrix* is a square matrix whose inverse is equal to its transpose i.e. $\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}}$.