Fact Sheet F
Directions, Lines and Planes

- A unit vector defining a particular direction may be written

  $$\hat{\mathbf{d}} = \frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k} = \ell\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k}$$

  where $\ell, m,$ and n are the *direction cosines* of $\hat{\mathbf{d}}$ as indicated. Note that

  $$\sqrt{\ell^2 + m^2 + n^2} = 1$$

- To find the direction cosines of a vector that is *not* a unit vector, divide by its magnitude to obtain a unit vector in the same direction (see Fact Sheet A). For example, for the vector $\mathbf{d} = 2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$, divide by $\sqrt{29}$ to obtain the unit vector $\hat{\mathbf{d}} = \frac{2}{\sqrt{29}}\mathbf{i} + \frac{4}{\sqrt{29}}\mathbf{j} + \frac{3}{\sqrt{29}}\mathbf{k}$. The coefficients of **d** (i.e. 2, 4 and 3) are called *direction ratios*, and the coefficients of $\hat{\mathbf{d}}$ are the *direction cosines* .

- The angle between two unit vectors can be found from

  $$\hat{\mathbf{d}}_1 . \hat{\mathbf{d}}_2 = \cos\theta = \ell_1\ell_2 + m_1 m_2 + n_1 n_2$$

  The angle between two arbitrary vectors **A** and **B** follows from

  $$\cos\theta = \hat{\mathbf{a}}.\hat{\mathbf{b}} = \frac{\mathbf{A}.\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{A_x B_x + A_y B_y + A_z B_z}{|\mathbf{A}||\mathbf{B}|} = \frac{A_x B_x + A_y B_y + A_z B_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}}$$

- There are several ways of writing the equation of *a straight line in 2D*. A line with gradient *g* and *y*-axis intercept *h* is given by

  $$y = gx + h$$

  A line passing through (x_0, y_0) in a direction defined by ℓ and m is given by

  $$\frac{x - x_0}{\ell} = \frac{y - y_0}{m}$$

  A line passing through $\mathbf{r}_0 = (x_0, y_0)$ in a direction $\hat{\mathbf{d}}$ is given by

  $$\mathbf{r} = \mathbf{r}_0 + \lambda\hat{\mathbf{d}}$$

  where λ is a variable parameter.

  A straight line can also be specified in the form

  $$ax + by = k$$

  In this case, *a* and *b* are the direction ratios of the *normal*, i.e. the vector drawn normal to the line from the origin. Divide all terms in the equation by $\sqrt{a^2 + b^2}$ to obtain

$$\underbrace{\frac{a}{\sqrt{a^2+b^2}}}_{\ell'} x + \underbrace{\frac{b}{\sqrt{a^2+b^2}}}_{m'} y = \frac{k}{\sqrt{a^2+b^2}} = p$$

The coefficients of x and y are now the direction cosines of the normal vector and p is the length of the normal i.e. the perpendicular distance from the origin to the line. Since the normal is (by definition) perpendicular to the line, it follows that

$$\ell\ell' + mm' = 0$$

- The *equation of a plane* in 3D is

  $$ax + by + cz = k$$

  where, by analogy with the case of a line in 2D above, a, b, and c are the direction ratios of the vector drawn normal to the plane from the origin. Divide through by $\sqrt{a^2+b^2+c^2}$ to obtain

  $$\underbrace{\frac{a}{\sqrt{a^2+b^2+c^2}}}_{\ell'} x + \underbrace{\frac{b}{\sqrt{a^2+b^2+c^2}}}_{m'} y + \underbrace{\frac{c}{\sqrt{a^2+b^2+c^2}}}_{n'} z = \frac{k}{\sqrt{a^2+b^2+c^2}} = p$$

  where p is now the perpendicular distance from the origin to the plane.

- The equation of *a straight line in 3D* is

  $$\frac{x-x_0}{\ell} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$$