Fact Sheet C – Complex Numbers 1

- A *complex number* is one that involves the factor $i = \sqrt{-1}$. (Note that *j* is used instead of *i* in some applications, e.g. electricity where *i* could be confused with current.)
- A general complex number *z* can be written in the standard form $z = x + iy$, where *x* and *y* are known respectively as the *real part* and the *imaginary part* of *z*.
- A complex number is represented geometrically by a point in the *Argand diagram* in which the real part is plotted on the *x*-axis and the imaginary part on the *y*-axis.
- Equally, one can use polar coordinates and write

  $$z = r(\cos\theta + i\sin\theta)$$

  where $r\cos\theta$ is the real part and $r\sin\theta$ is the imaginary part of *z*. The parameter *r* is called the *modulus* of *z* (basically its magnitude); it is written $|z|$, and referred to as "mod z". The angle θ is called the *argument* or *phase* of *z*.

  Clearly $|z| \equiv r = \sqrt{x^2 + y^2}$ and $\tan\theta = y / x$.
- It is easy (as in the lectures) to show that

  $$\boxed{\cos\theta + i\sin\theta = e^{i\theta}}$$

  The late Professor Richard Feynman said this was "*the most remarkable formula in mathematics*", linking trigonometry on the lhs with algebra on the rhs. It follows that

  $$\boxed{z = re^{i\theta}}$$
- The *complex conjugate* of a complex number is obtained by reversing the sign of the imaginary part, which can be done by replacing *i* with –*i* everywhere it appears. Thus, if $z = x + iy = r\,\mathrm{e}^{i\theta}$, then $z^* = x - iy = r\,\mathrm{e}^{-i\theta}$. Note the use of * to indicate complex conjugation.
- Complex numbers can be added, subtracted, multiplied, and divided in the normal way using either the (*x*, *y*) or the (*r*, *θ*) form. Just remember to keep track of the factor *i*. Hence, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

  The square of *i* can of course be replaced by –1 so, for example, $i^3 \equiv i \times i^2 = -i$ etc. Hence $z^2 = (x^2 - y^2) + 2ixy = r^2 e^{2i\theta}$ and $zz^* \equiv |z|^2 = r^2 = x^2 + y^2$. Notice that multiplying a complex number by its complex conjugate yields the square of the modulus.

  A useful trick to facilitate the division of one complex number by another is to multiply top and bottom by the complex conjugate of the denominator e.g.

  $$\frac{4+5i}{2-3i} = \frac{4+5i}{2-3i} \times \frac{2+3i}{2+3i} = \frac{-7+22i}{13} = -\frac{7}{13} + i\frac{22}{13}.$$