Classwork 8 *(update 2)*

A Christmas Medley

In this last Classwork, you have a choice of what to do:

- If you want to enter into the spirit of the festive season, try question 1.
- If you'd like to do a very simple exercise with complex numbers, inherited from one of last year's Classworks, try question 2.
- If you would prefer a serious question on eigenvalues and eigenvectors, try number 3.

Or of course do all of them in sequence. Whatever you decide, I hope you have a happy Christmas!

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1. Santa's sleigh is approaching Imperial College, which is of course the origin of all right-minded (and right-handed) coordinate systems. Santa is heading in a southwesterly direction (having come from Lapland), and is descending steeply on the track $\mathbf{r} = \lambda(-\mathbf{i} - \mathbf{j} - \mathbf{k})$ where x is east, y is north, and z is the upwards vertical.

The wicked witch/Darth Vader/Tash/Voldemort/Sauron ……….................…. (enter your choice) is rising rapidly from the depths on the path $\mathbf{r} = -750(\mathbf{j} + \mathbf{k}) + \mu(4\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})$ to intercept Santa and steal your presents!

Find the distance of closest approach of the two tracks. Units are à choix – make them at least kilometres to ensure that Santa arrives safely, but light years would be even better!

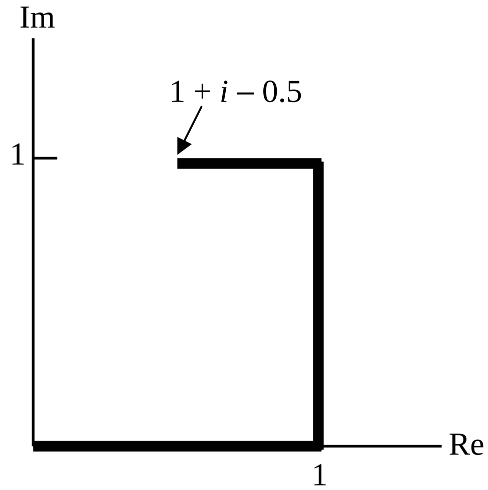
Hint: All the information you need to do this question is reproduced overleaf.

2. MacLaurin's expansion of e^x, gives $e^i = 1 + \dfrac{i}{1!} + \dfrac{i^2}{2!} + \dfrac{i^3}{3!} + ... = 1 + i + 0.5i^2 + 0.16667i^3 + ...$

(i) Simplify the third and fourth terms in the equation.

(ii) Write out the next four terms to five decimal places, and simplify as appropriate.

(iii) The figure illustrates the start of the expansion. Starting from the origin, each term is added in succession. Continue the figure by adding the next five terms.



(iv) It will be clear that the series is converging to a definite point in the complex plane which represents the complex number e^i. Calculate the real part, the imaginary part, the modulus, and the argument (in radians) of this complex number (all to four decimal places). Of course, having been to the lectures, you'll know what the answers are anyway!

3. Find the eigenvalues and eigenvectors of the matrices

(i) $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

Hint: Un-normalised eigenvectors will do. In (ii) and (iii), you will have to solve a cubic equation for the eigenvalues. However, the eigenvalues are small integers in both cases so, once you have found one of them (λ_1) by inspection, you can divide through by $(\lambda - \lambda_1)$ and solve a quadratic for the other two.

<u>How to do question 1</u>

The shortest distance between two straight line paths is $s = |\mathbf{V}.\hat{\mathbf{n}}|$ where **V** is <u>*any*</u> vector linking points on the two paths, and $\hat{\mathbf{n}}$ is the unit normal to both of them. To get $\hat{\mathbf{n}}$, use the fact that it is perpendicular to the two direction vectors $\mathbf{d_1}$ and $\mathbf{d_2}$, which appear in the equations for the two tracks. The formula you need is $\hat{\mathbf{n}} = \dfrac{\mathbf{d_1} \times \mathbf{d_2}}{|\mathbf{d_1} \times \mathbf{d_2}|}$.

<u>Updates</u>

- The original version had an error in the formula for $\hat{\mathbf{n}}$ at the end of the previous paragraph; this was corrected by hand on most if not all photocopies.
- The original version also gave the line of interception in question 1 as

  $\mathbf{r} = -750(\mathbf{j}+\mathbf{k}) + \mu(2\mathbf{i}+5\mathbf{j}+3\mathbf{k})$. For this vector, the answer sheet should read:

  " ………………. It follows that $\hat{\mathbf{n}} = \dfrac{-\mathbf{i}-\mathbf{j}-\mathbf{k}}{\sqrt{3}} \times \dfrac{2\mathbf{i}+5\mathbf{j}+3\mathbf{k}}{\sqrt{38}} = \dfrac{2\mathbf{i}+\mathbf{j}-3\mathbf{k}}{\sqrt{114}}$. A convenient choice for **V** is $\mathbf{V} = -750(\mathbf{j}+\mathbf{k})$, and from the dot product, one obtains $s = \dfrac{1500}{\sqrt{114}} = 140.5$, a comfortable margin if the units are km. …………… "