Classwork 6
Complex Oscillations

Note: Complex quantities are identified throughout by a tilde e.g. $\tilde{x}, \tilde{A}$.

In this classwork, we take a break from vectors and matrices, and pick up some ideas from Lecture 3 on the exponential form of complex numbers.

Problems involving oscillations are often solved using complex numbers. In this kind of application, the physical parameters are themselves all real, but complex numbers make the mathematics easier. Consider for instance a mass on a spring, which is displaced from its equilibrium position and allowed to oscillate. We are interested in $x(t)$, the displacement of the mass from equilibrium as a function of time. Although $x(t)$ is itself real, it is convenient to treat it as the real part of a complex "displacement" $\tilde{x}(t)$, because this makes it easier to solve the equations. Once the problem is solved for $\tilde{x}(t)$, one takes the real part of the answer to get back to $x(t)$; see note at end for more information.

This classwork introduces you to the use of complex numbers for representing oscillations.

(a) The position of an object as a function of time is given by the real part of $\tilde{x}(t) = Ae^{i\tilde{\omega}t}$ where A and $\tilde{\omega}$ are constants. Sketch the motion if (i) $\tilde{\omega} = \omega$ (real) and (ii) $\tilde{\omega}$ is pure imaginary.

(b) If $\tilde{x} = \tilde{A}e^{i\omega t}$ where $\tilde{A} = Ae^{i\theta}$, sketch the motion when (i) $\theta = -\pi/2$ and (ii)) $\theta = \pm\pi$.

(c) If $\tilde{\omega}$ is complex, i.e. $\tilde{\omega} = \omega + i\gamma$, the motion has the form of damped oscillations in which the amplitude of the oscillations decays exponentially with time. Write down expressions for (i) x at t = 0, (ii) the oscillation period T (remember that frequency = angular frequency ÷ 2π), and (iii) the time τ taken for the amplitude of the oscillations to decay to half its value.

(d) Ignoring friction, Newton's second law applied to an object of mass m attached to a spring of spring constant k is $m\frac{d^2x}{dt^2} = -kx$. Here x represents displacement from equilibrium, and the right hand side is the restoring force proportional to the displacement (the – sign arises because it is a *restoring* force).

This problem can be solved by using the complex analogue of the previous equation namely $m\frac{d^2\tilde{x}}{dt^2} = -k\tilde{x}$ which has solutions of the form $\tilde{x}(t) = \tilde{A}e^{i\omega_0 t}$. Substitute this into the differential equation to find the natural angular frequency ω_0 in terms of *m* and *k*.

(e) Write down an expression for the speed $\tilde{v}$ of the object in complex form.

(f) In the case where $\tilde{A} = A$ (real), take the real part of the complex variables to show that the physical displacement and speed at time *t* are $x = A\cos\omega_0 t$ and $v = -A\omega_0 \sin\omega_0 t$.

(g) For *m* = 4 kg, *k* =100 N/m and *A* = 0.1 m, find (i) the angular frequency ω_0, (ii) the period of oscillation, and (iii) the displacement and speed at *t* = 0, 0.2, and 0.4 s.

(h) Repeat part (f) for the case where $\tilde{A} = Ae^{i\pi/2}$.

If time permits ….

(i) Now play the same game with the equation $m\frac{d^2\tilde{x}}{dt^2} = -k\tilde{x} - \rho\frac{d\tilde{x}}{dt}$, which includes a frictional (damping) term. Substitute $\tilde{x}(t) = \tilde{A}e^{i\tilde{\omega}t}$ into the equation and solve for $\tilde{\omega}$.

<u>The advantage of complex numbers for treating oscillatory systems.</u>

Complex numbers make this kind of problem easier because $e^{i\omega t}$ is simpler to handle than $\cos\omega t$ and $\sin\omega t$. It differentiates into itself (apart from a factor) because $\frac{d(e^{i\omega t})}{dt} = i\omega e^{i\omega t}$, it combines naturally with exponential decay and growth, and a phase change is readily introduced by multiplying $e^{i\omega t}$ by $e^{i\theta}$ to yield $e^{i(\omega t+\theta)}$.