Classwork 5
Discover the Orthogonal Matrix

A unit vector is often called a normalised vector. Two general vectors that are perpendicular to each other are said to be "orthogonal", and two unit vectors that are perpendicular to each other are often said to be "orthonormal". This classwork, which consists of many short parts, leads you on, in gentle steps, to the definition of an orthogonal matrix. Although the questions relate to two-dimensional vectors and 2×2 matrices, the results are general.

(a) Show that two two-dimensional vectors $\mathbf{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ are orthonormal if $u_x^2 + u_y^2 = v_x^2 + v_y^2 = 1$ and $u_x v_x + u_y v_y = 0$. Satisfy yourself that these conditions can be expressed in the matrix form $\mathbf{v}^{\mathrm{T}}\mathbf{v} = \mathbf{u}^{\mathrm{T}}\mathbf{u} = 1$ and $\mathbf{v}^{\mathrm{T}}\mathbf{u} = \mathbf{u}^{\mathrm{T}}\mathbf{v} = 0$.

(b) Find the unit vector in the direction of $\mathbf{a} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, and find two other vectors $\hat{\mathbf{b}}$ that are orthonormal to $\hat{\mathbf{a}}$.

(c) Consider the 2×2 matrix made up of the two orthonormal vectors from part (a) namely $\mathbf{M} = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}$. Show that $\mathbf{M}^{\mathrm{T}}\mathbf{M} = \mathbf{U}$ where $\mathbf{U}$ is the unit (or identity) matrix. A matrix of this kind is known as an *orthogonal matrix*.

(d) We will now discover some more properties of orthogonal matrices. Consider two vectors $\mathbf{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} q_x \\ q_y \end{pmatrix}$ related by $\mathbf{q} = \mathbf{N}\mathbf{p}$ where $\mathbf{N} = \begin{pmatrix} c & d \\ e & f \end{pmatrix}$. If the magnitudes of **p** and **q** are the same, what conditions are imposed on the elements of **N**?

(e) If $\mathbf{q} = \mathbf{N}\mathbf{p}$, which of the following is correct: (i) $\mathbf{q}^{\mathbf{T}} = \mathbf{N}^{\mathbf{T}}\mathbf{p}^{\mathbf{T}}$ or (ii) $\mathbf{q}^{\mathbf{T}} = \mathbf{p}^{\mathbf{T}}\mathbf{N}^{\mathbf{T}}$?

(f) Consider two vectors **p** and **g** which are transformed by the matrix **N** into **q** and **h** so that **q = Np** and **h = Ng** respectively. If the transformation does not change the scalar product (i.e. **g.p = h.q** or in matrix form $\mathbf{g}^{\mathrm{T}}\mathbf{p} = \mathbf{h}^{\mathrm{T}}\mathbf{q}$), show that $\mathbf{N}^{\mathrm{T}}\mathbf{N} = \mathbf{U}$ in other words that **N** is orthogonal. The situation in part (d) is the special case where **g** = **p** and **h** = **q**.

(g) Is the rotation matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ orthogonal?

(h) Put $\hat{\mathbf{a}}$ and each of the two $\hat{\mathbf{b}}$ vectors from part (b) in turn together to form two orthogonal matrices $\mathbf{L}_1$ and $\mathbf{L}_2$ like **M** in part (c). If the transformation represents a rotation, find the angle. If not, try to figure out what the operation does represent.

(i) Transform the vector $\mathbf{s} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$ using each orthogonal matrix from part (h) in turn. Check that the new vectors $\mathbf{t}_1 = \mathbf{L}_1\mathbf{s}$ and $\mathbf{t}_2 = \mathbf{L}_2\mathbf{s}$ have the same magnitude as **s**. Find the angle between **s** and each new vector, and draw all three vectors on a diagram.